# On the spectral geometry of manifolds with conic singularities

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# Contents

# 1 Introduction

Consider a Riemannian manifold, $(M, g)$, of dimension $m$. The Laplace-Beltrami operator is, by definition, the Hodge Laplacian restricted to smooth functions on $(M, g)$. The space of smooth functions can be completed to the Hilbert space of square integrable functions. The Laplace-Beltrami operator, $\Delta$, is a symmetric non-negative operator and it always has a self-adjoint extension, the Friedrichs extension. We are interested in those Riemannian manifolds where the Friedrichs extension of the Laplace-Beltrami operator has discrete spectrum, $\operatorname{spec}\Delta$, see Section 2.1.

Spectral geometry studies the relationship between the geometry of $(M, g)$ and $\operatorname{spec}\Delta$. One of the main tools of spectral geometry is the *heat trace*

$$\operatorname{tr} e^{-t\Delta} = \sum_{\lambda \in \operatorname{spec}\Delta} e^{-t\lambda}. \tag{1.1}$$

For compact Riemannian manifolds $(M, g)$, the problem of finding geometric information from the eigenvalues of the Laplace-Beltrami operator and the Hodge Laplacian has been extensively studied, see e.g. [G2] and the references given there. On closed $(M, g)$ there is an asymptotic expansion

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to+0} (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} a_j t^j, \tag{1.2}$$

where $a_j \in \mathbb{R}$. In principle, every term in (1.2) can be written as an integral over the manifold of a local quantity. Namely,

$$a_j = \int_M u_j \, \mathrm{dvol}_M, \tag{1.3}$$

where $u_j$ is a polynomial in the curvature tensor and its covariant derivatives, see Section 3.1. In particular, $u_0 = 1$ and $u_1 = \frac{1}{6}\,\mathrm{Scal}$, where Scal is the scalar curvature of $(M, g)$. The bigger $j$, the more complicated the calculation of $u_j$. Sometimes we write $u_j(p)$ to indicate that it is a local quantity, i.e. it depends on a point $p \in M$.

There are many examples of manifolds that are *isospectral*, i.e. have the same spectrum of $\Delta$, but are not isometric, see the survey [GPS]. However, it remains very interesting to study to what extent the geometry of $(M, g)$ can be determined from $\operatorname{spec}\Delta$.

In this thesis we study spectral geometry on a non-complete smooth Riemannian manifold $(M, g)$ that possesses a conic singularity. By this we mean that there is an open subset $U$ such that $M \setminus U$ is a smooth compact manifold with boundary $N$. Furthermore, $U$ is isometric to $(0, \varepsilon) \times N$ with $\varepsilon > 0$, where the *cross-section* $(N, g_N)$ is a closed smooth manifold, and the metric on $(0, \varepsilon) \times N$ is

$$g_{\text{conic}} = dr^2 + r^2 g_N, \quad r \in (0, \varepsilon). \tag{1.4}$$

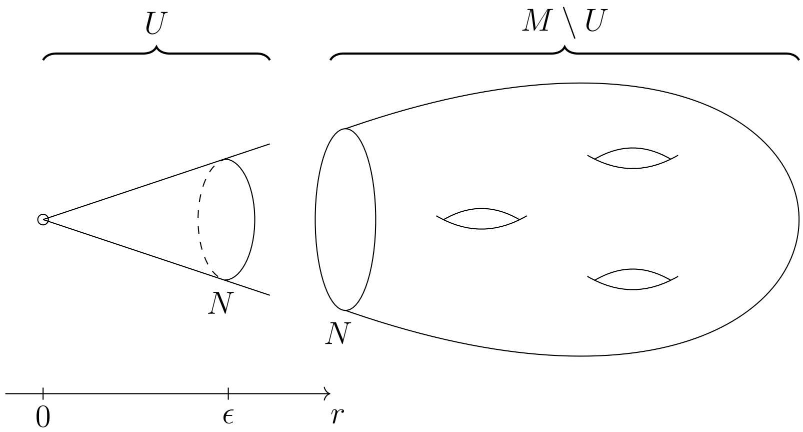


Figure 1: Manifold with a conic singularity.

For a particular choice of $g_N$, the conic metric $g_{\text{conic}}$ provides an isometry with the punctured ball. Namely, let $(N, g_N)$ be a unit sphere with the round metric, then $U$ is isometric to a punctured ball with the metric $g_{\text{conic}}$. If we include the conic point to the neighbourhood $\bar{U} := [0, \varepsilon) \times N$, we see that there is no singularity, but rather we have polar coordinates in the neighbourhood $\bar{U}$. Informally speaking a conic singularity is not necessarily a singularity, it will then be referred to as the apparent singularity. We illustrate this in the two-dimensional case on Figure 2. The metric on $U$ is $g_{\text{conic}} = dr^2 + r^2 \sin^2 \alpha d\theta^2$, where the parameter $0 < \alpha \leq \pi/2$ denotes the angle between the generating line and the axis of the cone and $0 < \theta \leq 2\pi$ is the coordinate on $S^1$.

The existence of the heat trace expansion of the Friedrichs extension of the Hodge Laplacian on the differential forms on manifolds with conic singularities was proven by Jeff Cheeger [Ch, Section 5]. Jochen Brüning and Robert Seeley in [BS] and [BS2] developed a general method for showing the existence of the heat trace expansion of second order elliptic differential operators. A fundamental feature of the expansion on manifolds with conic

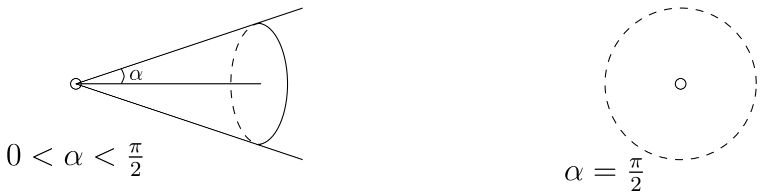


Figure 2: Actual singularity vs apparent singularity.

singularities is that a logarithmic term can appear (see also [BKD, (4.6)]), while only power terms can appear in (1.2) on a smooth closed manifold. It was not fully understood how a singularity contributes to the coefficients in the expansion. In this thesis we used the local heat kernel expansion, see Section 3.1, and then the Singular Asymptotics Lemma from [BS2, p. 372], see Section 2.4, to compute the terms in the heat trace expansion on a manifold with a conic singularity.

The negative power terms in the expansion do not have any contribution from the singularity and are computed for a bounded cone with different boundary conditions by Michael Bordag, Klaus Kirsten and Stuart Dowker in [BKD, (4.7)–(4.8)]. The first power term in the expansion that is affected by the singularity is the constant term. The expression of the constant term in [Ch2, Theorem 4.4] and [BKD, (4.5)] involves residues of the spectral zeta function of the Laplace-Beltrami operator as well as the finite part of the spectral zeta function at a particular point $s \in \mathbb{C}$. In a more general setup in [BS2, (7.22)] the constant term is expressed as the infinite sum of residues of the spectral zeta function of a certain operator on $(N, g_N)$ plus the analytic continuation of the zeta function at a particular point. Here we show that the sum in the expression of the constant term is finite for the case of conic singularities.

Since the manifold $(M, g)$ is non-compact, it may happen that the Laplace-Beltrami operator on $(M, g)$ has many self-adjoint extensions. To be able to apply the Singular Asymptotics Lemma we need the operator to satisfy the scaling property, see Section 3.3. It is known that the Friedrichs extension has this property and we restrict our attention to this particular self-adjoint extension. We now present the main theorem.

**Theorem 1.1.** *Let $\Delta$ be the Laplace-Beltrami operator on smooth functions with compact support on $(M, g)$. If $m \geq 4$, then $\Delta$ is essentially self-adjoint operator, otherwise we consider the Friedrichs extension of $\Delta$. Denote the self-adjoint extension of the Laplace-Beltrami operator by the same symbol $\Delta$.*

*Then*

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to 0+} (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} \tilde{a}_j t^j + b + c \log t, \tag{1.5}$$

*(a) where*

$$\tilde{a}_j = \begin{cases} \int_M u_j \operatorname{dvol}_M & \textit{for } j \le m/2 - 1, \\ ⨍_M u_j \operatorname{dvol}_M & \textit{for } j > m/2 - 1. \end{cases}$$

*Above* $⨍$ *denotes the regularized integral, which we define in Section 2.4, of local quantities* $u_j$ *in (1.3).*

*(b) The constant term* $b$ *in general cannot be written in terms of local quantities, and is given by*

$$\begin{aligned} b = &-\frac{1}{2} \operatorname{Res}_0 \zeta_N^{\frac{m-2}{2}}(-1/2) + \frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} \operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(-1/2) \\ &- \frac{1}{4} \sum_{j=1}^{m/2} j^{-1} B_{2j} \operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(j - 1/2), \end{aligned}$$

*where* $\zeta_N^l(s) = \sum_{\lambda \in \operatorname{spec}\Delta_N} (\lambda + l^2)^{-s}$ *is the spectral zeta function shifted by* $l$*. The constants* $B_{2j}$ *are the Bernoulli numbers,* $\operatorname{Res}_0 f(s_0)$ *is the regular analytic continuation of a function* $f(s)$ *at* $s = s_0$*, and* $\operatorname{Res}_1 f(s_0)$ *is the residue of the function* $f(s)$ *at* $s = s_0$*.*

*(c) The logarithmic term is given by*

$$c = \begin{cases} \frac{1}{4(4\pi)^{\frac{m}{2}}} \sum_{k=0}^{\frac{m}{2}} (-1)^{k+1} \frac{(m-2)^{2k}}{4^k k!} a^N_{\frac{m}{2}-k}, & \textit{for } m \textit{ – even,} \\ 0, & \textit{for } m \textit{ – odd.} \end{cases} \tag{1.6}$$

*(d) If* $c = 0$ *then* $\tilde{a}_{m/2} = \int_M u_{m/2} \operatorname{dvol}_M$ *does not have a contribution from the singularity.*

*Here* $\Delta_N$ *is the Laplace-Beltrami operator on the cross-section* $(N, g_N)$ *and* $a_j^N, j \ge 0$ *denote the coefficients in the heat trace expansion (1.2) on* $(N, g_N)$*.*

Above we need to regularize the integrals, because in general $\int_M u_j \operatorname{dvol}_M$ diverges. If for some $j \ge 0$ the integral converges, i.e. $⨍_M u_j \operatorname{dvol}_M = \int_M u_j \operatorname{dvol}_M$, then in this case $\tilde{a}_j$ is equal to $a_j$ from (1.3).

Theorem 1.1 allows to connect the coefficients in (1.5) to the geometry of $(M, g)$. It is now natural to pose the following question: given the coefficients in (1.5), can we say if there are actual or only apparent singularities? The idea is to compare the expansion (1.5) to the expansion on a smooth compact manifold (1.2). In case $c \neq 0$ we have an actual singularity, whereas if $c = 0$ we need to compute $b$ to detect a singularity.

Consider now the case of even-dimensional manifolds. While the logarithmic term $c$ is written in terms of the geometry of the cross-section near the singularity, the constant term $b$ is expressed in residues and regular values of the spectral zeta function of the cross-section. Thus it is difficult to extract the geometric meaning of $b$ in general; therefore, we study low dimensions one by one.

The heat trace expansion for the Friedrichs extension of the Laplace-Beltrami operator on the algebraic curves was developed by Brüning and Lesch in [BL, Theorem 1.2]. In general, many logarithmic terms $c_j t^j \log t$ with $j \geq 0$ might appear in the expansion. We prove that in the heat trace expansion on a surface with conic singularities there is no logarithmic term and the geometric information about the singularities is encoded only in the constant term. In this case the spectral zeta function on the cross-section $(N, g_N)$ is the Riemann zeta function, so the computations can be done explicitly.

**Lemma 1.2.** *Let $(M, g)$ be a surface with $l$ conic singularities. Let $\Delta$ be the Friedrichs extension of the Laplace-Beltrami operator. Then*

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to 0+} \frac{1}{4\pi t}\sum_{j=0}^{\infty} a_j t^j + \frac{1}{12}\sum_{i=1}^{l}\left(\frac{1}{\sin\alpha_i} - \sin\alpha_i\right), \tag{1.7}$$

*where $\alpha_i$ is the angle between the generating line and the axis of the cone corresponding to the $i$-th conic singularity. Above $a_j$ does not have any contribution from the singularities for $j \geq 0$.*

Denote by $(\bar{M}, g)$ the complete surface with conic points included, i.e. the neighbourhoods of conic points are $U' = [0, \varepsilon) \times N$. By the expansion in Lemma 1.2, we obtain the following.

**Theorem 1.3.** *Let $(\bar{M}, g)$ be a complete simply-connected surface with conic singularities. If $(\bar{M}, g)$ has at least one singularity, then it is not isospectral to any smooth closed surface.*

In even dimensions $m \geq 4$ the cross-section $(N, g_N)$ is a closed manifold of dimension three or more. The spectral zeta function of $N$ is known explicitly only in very few cases which makes the situation in dimensions $m \geq 4$ much more complicated. We present the spectral geometry of a four-dimensional $(M, g)$.

**Theorem 1.4.** *Let $(M, g)$ be a four-dimensional manifold with conic singularities.*

*(1) The logarithmic term in the heat trace expansion (1.5) is equal to zero if and only if the cross-section of every singularity is isometric to a spherical space form.*

*(2) Assume that the logarithmic term in the heat trace expansion (1.5) is equal to zero. Then $\tilde{a}_j = a_j$ for $j \geq 0$. In this case only the constant term in the heat trace expansion has a contribution from the singularities.*

Denote by $(\bar{M}, g)$ the manifold with conic singularities with conic points included.

**Corollary 1.5.** *Let $(\bar{M}, g)$ be a complete four-dimensional manifold with conic singularities. If $(\bar{M}, g)$ has at least one singularity with a cross-section $(N, g_N)$ not isometric to a spherical space form, then it is not isospectral to any smooth compact four-dimensional manifold.*

At some point there was a hope that a theorem similar to Theorem 1.4 holds true for any even-dimensional manifold with conic singularities, so we determine a criterion for the logarithmic term in the heat trace expansion to vanish.

**Lemma 1.6.** *Let $(M, g)$ be a even-dimensional manifold with a conic singularity. The logarithmic term in the heat trace expansion on $(M, g)$ is equal to zero if and only if the following equality holds for the heat trace coefficients of the $n$-dimensional cross-section manifold $(N, g_N)$*

$$a^N_{\frac{n+1}{2}} = \sum_{k=1}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} a^N_{\frac{n+1}{2}-k}. \tag{1.8}$$

*Remark.* In the $n = 1$ case this equality is always true. In the $n = 3$ case this equality implies that the sectional curvature of the cross-section manifold $(N, g_N)$ is constant $\kappa = 1$ (by Theorem 1.4). Let us analyse what geometric restrictions we obtain from this equality in higher dimensional cases.

**Theorem 1.7.** *Assume that the cross-section manifold $(N, g_N)$ has constant sectional curvature $\kappa$. Then the logarithmic term in the heat trace expansion on $(M, g)$ can be written as the polynomial in $\kappa$ of degree $\frac{n+1}{2}$*

$$c = \frac{1}{8\sqrt{\pi}} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)} \sum_{k=0}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} \sum_{l=1}^{\frac{n-1}{2}} \frac{(\frac{n-1}{2})^{2l-2k+2} \Gamma(l+\frac{1}{2}) K_l^{\frac{n-1}{2}}}{(l-k+1)!(n-1)!} \kappa^{\frac{n+1}{2}-k},$$

*where numbers $K_l^{\frac{n-1}{2}}$ are given by (5.1) and depend only on $n$ and $l$.*

The next results show that Theorem 1.4 cannot be extended to higher dimensions.

**Corollary 1.8.** *Let $(N, g_N)$ be a five-dimensional manifold with constant sectional curvature $\kappa$. The logarithmic term in the heat trace expansion on $(M, g)$ is zero if and only if $\kappa = 1$ or $\kappa = 2$.*

**Corollary 1.9.** *Let $(N, g_N)$ be a seven-dimensional manifold with constant sectional curvature $\kappa$. The logarithmic term in the heat trace expansion on $(M, g)$ is zero if and only if $\kappa = 1$ or $\kappa = \frac{225}{109} \pm \frac{36\sqrt{5}}{109}$.*

From the above results we conclude that the higher the dimension of the manifold $(M, g)$, the less geometric information we can obtain from the heat trace expansion on $(M, g)$. If $(\bar{M}, g)$ is a complete simply-connected surface with conic singularities, from the heat trace expansion on $(M, g)$ we can determine whether $(M, g)$ has an actual singularity or an apparent singularity. If $(M, g)$ is a four-dimensional manifold with conic singularities, we can determine whether a cross-section of the singularity is isometric to a spherical space form. If $(M, g)$ is a higher dimensional manifold, the situation becomes less determined.

This thesis is organized as follows. In Section 2, we present a geometric setup, then define regularized integrals following [L, Section 2.1], and state the main lemma from [BS2, p. 372], the Singular Asymptotics Lemma. In Section 3, we prove that the conditions of the Singular Asymptotics Lemma are satisfied in our case. We then apply it to the expansion of the trace of the resolvent. In Section 3.5, we compute the coefficients in the heat trace expansion for a manifold with conic singularities. In Section 3.6, we assemble the proof of Theorem 1.1. In Section 4, we prove Lemma 1.2, Theorem 1.3, Theorem 1.4 and Lemma 1.6. In Section 5, we prove Theorem 1.7, Corollary 1.8 and Corollary 1.9. We conclude with the computations of the logarithmic term and the constant term for some particular $n$-dimensional cross-sections.

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# 2 Preliminaries

## 2.1 The Laplace-Beltrami operator on an open manifold

In this section we present some basic notions and theorems about operators on Hilbert spaces, following [W]. Then we show how this results apply to the Laplace-Beltrami operator on a manifold with conic singularity.

Let $H_1$, $H_2$ be Hilbert spaces. Let $A$ be an operator from $H_1$ to $H_2$, and $B$ be an operator from $H_2$ to $H_1$. The operator $B$ is called a *formal adjoint* of $A$ if we have

$$\langle h, Ag\rangle = \langle Bh, g\rangle \text{ for all } g \in D(A), h \in D(B),$$

where $D(A), D(B)$ are the domains of the operators $A, B$. We denote formal adjoint of $A$ by $A^\dagger$.

Let $A$ be an operator on a Hilbert space $H$. The operator $A$ is called *symmetric* if for any elements $h, g$ from its domain we have $\langle h, Ag\rangle = \langle Ah, g\rangle$. A densely defined symmetric operator $A$ is called *self-adjoint* if it is equal to its adjoint $A = A^\dagger$ and *essentially self-adjoint* if its closure is equal to its adjoint. An operator $B$ is called an *extension* of $A$ if we have

$$D(A) \subset D(B) \text{ and } Ah = Bh \text{ for } h \in D(A).$$

If $A$ is a symmetric operator, then $A \subset A^\dagger$, [W, p.72]. For every symmetric extension $B$ of $A$ we have $A \subset B \subset B^\dagger \subset A^\dagger$. If $B$ is self-adjoint, then $A \subset B = B^\dagger \subset A^\dagger$.

A symmetric operator $A$ on the Hilbert space $H$ is said to be *bounded from below* if there exists $a \in \mathbb{R}$ such that $\langle h, Ah\rangle \geq a\|h\|^2$ for all $h \in D(A)$. Every $a$ of this kind is called a *lower bound*. If zero is a lower bound of $A$, then $A$ is called *non-negative*.

**Theorem 2.1** (Friedrichs extension, [W, Theorem 5.38])**.** *A non-negative densely defined symmetric operator $A$ on a Hilbert space $H$ has a non-negative self-adjoint extension.*

We consider a non-complete smooth Riemannian manifold $(M, g)$ that possesses a conic singularity, i.e. there is an open subset $U$ such that $M \setminus U$ is a smooth compact manifold with boundary $N$. Furthermore, $U$ is isometric to $(0, \varepsilon) \times N$ with $\varepsilon > 0$, where the *cross-section* $(N, g_N)$ is a closed smooth manifold, and the metric on $(0, \varepsilon) \times N$ is

$$g_{\text{conic}} = dr^2 + r^2 g_N, \quad r \in (0, \varepsilon). \tag{2.1}$$

Consider the space of smooth functions with compact support $C_c^\infty(M)$ on $(M, g)$, and the space of differential one-forms with compact support $\lambda_c^1(M) := C_c^\infty(\Lambda T^*M)$. There are certain first order differential operators defined between these spaces, exterior derivative $d : C_c^\infty \to \lambda_c^1(M)$ and its formal adjoint $d^\dagger = - * d* : \lambda_c^1(M) \to C_c^\infty(M)$, where $*$ is the Hodge-star operator on the differential forms on $(M, g)$. Furthermore, the operator $\Delta := d^\dagger d$, defined on the smooth functions with compact support, is called the Laplace-Beltrami operator on $(M, g)$.

**Proposition 2.2.** *The operator $\Delta$ is densely defined in $L^2(M)$, symmetric and non-negative.*

*Proof.* Since $C_c^\infty(M)$ is dense in $L^2(M)$, the operators $d$ and $d^\dagger$ are densely defined respectively in $L^2(M)$ and $L^2(\Lambda T^*M)$. Hence $\Delta$ is densely defined in $L^2(M)$. Let $f \in C_c^\infty(M)$, then

$$(\Delta f, f) = (d^\dagger df, f) = (df, df) = (f, \Delta f) \geq 0.$$

From this follows that $\Delta$ is symmetric and non-negative. $\square$

By Theorem 2.1, the operator $\Delta$ admits a self-adjoint extension. In Section 3.6, we observe that for $\dim M = m < 4$ there can be many self-adjoint extensions of $\Delta$, if this is the case, we choose the Friedrichs extension $\Delta^F$, which we denote simply by $\Delta$. The reason we choose the Friedrichs extension is that it satisfies the scaling property, Section 3.3, which we need in Lemma 3.5.

In the next sections, we discuss an asymptotic expansion of the heat trace of the Laplace-Beltrami operator. For this purpose we deal with the operator separately on a neighbourhood $(U, g_{\text{conic}})$ and on the regular part $(M \setminus U, g)$. For the restriction $\Delta|_{M\setminus U}$ of the Laplace-Beltrami operator $\Delta$ to the regular part, we use the methods applicable for a compact manifold. As for the restriction $\Delta|_U$, we first extend $(U, g_{\text{conic}})$ to an infinite cone $((0, +\infty) \times N, g_{\text{conic}})$, then extend the Laplace-Beltrami operator to the infinite cone and multiply the restriction $\Delta|_{(0,+\infty)\times N}$ by a function with the support near the tip of a cone and use the Singular Asymptotics Lemma, Lemma 2.7. To glue the result on the infinite cone and the result on the regular part, we use a partition of unity. We observe that the heat trace expansion does not depend on $\varepsilon$ in (2.1).

## 2.2 Spaces $L^2((0, \varepsilon) \times N)$ and $L^2((0, \varepsilon), L^2(N))$

In this subsection we introduce spaces that we consider in Section 3, and construct a bijective unitary map (2.3) between these spaces.

Let $E$ be a vector bundle over a smooth manifold $M$. Denote by $C^\infty(M, E)$ space of smooth sections of $E$ over $M$.

Let $I := (0, \varepsilon)$, where $0 < \varepsilon \leq +\infty$, and let $X$ be any set. Define

$$C^\infty(I, X) := \{\varphi : I \to X \mid \varphi \text{ is smooth }\}. \tag{2.2}$$

Consider a manifold $N$ and a projection map

$$\pi_N : I \times N \to N.$$

**Lemma 2.3.** *Let $G$ be a vector bundle over $N$. Then a section of pull-back bundle $\pi_N^* G$ at every $r \in I$ is a section of $G$, i.e. the following spaces are isomorphic*

$$C^\infty(I \times N, \pi_N^* G) \simeq C^\infty(I, C^\infty(N, G)).$$

*Proof.* We have the diagram

$$\begin{array}{ccc} \pi_N^* G & \longrightarrow & G \\ \downarrow & & \downarrow \\ I \times N & \xrightarrow{\pi_N} & N \end{array}$$

Let $(U_i, \tau_{ij})$ be a covering of $N$ by open sets $U_i$ such that the bundle $G$ restricted to $U_i$ is trivial $G|_{U_i} \simeq U_i \times \mathbb{R}^k$. Maps $\tau_{ij}$ are corresponding transition maps, i.e. smooth maps $\tau_{ij} : U_i \cap U_j \to GL(k)$, where $k$ is the rank of the bundle $G$. Then $(I \times U_i, \tau_{ij} \circ \pi_N)$ is a covering for the pull-back bundle $\pi_N^* G$.

Let $s \in C^\infty(I \times N, \pi_N^* G)$. Restrict the section $s$ on a chart

$$s|_{I \times U_i} : I \times U_i \to I \times U_i \times \mathbb{R}^k.$$

Since $C^\infty(I \times U_i, I \times U_i \times \mathbb{R}^k) \simeq C^\infty(I \times U_i, \mathbb{R}^k)$, we may reason in terms of maps. By the exponential law for smooth maps [KM, Theorem 3.12, Corollary 3.13], we have

$$C^\infty(I \times U_i, \mathbb{R}^k) \simeq C^\infty(I, C^\infty(U_i, \mathbb{R}^k)).$$

Above $C^\infty(\cdot, \cdot)$ denotes a space of the smooth maps as in (2.2). Now we pass back to the space of the smooth sections

$$C^\infty(I, C^\infty(U_i, \mathbb{R}^k)) \simeq C^\infty(I, C^\infty(U_i, U_i \times \mathbb{R}^k)),$$

and obtain the isomorphism of spaces

$$C^\infty(I \times U_i, I \times U_i \times \mathbb{R}^k) \simeq C^\infty(I, C^\infty(U_i, U_i \times \mathbb{R}^k)).$$

This isomorphism holds for any chart. Since the transition maps are smooth, we obtain the desired isomorphism. $\square$

Let $(N, g_N)$ be a closed smooth Riemannian manifold. Denote space of the differential $k$-forms on $(N, g_N)$ by $\lambda^k(N)$.

**Lemma 2.4.** *The following spaces are isomorphic*

$$\lambda^k(I \times N) \simeq C^\infty(I, \lambda^k(N) \oplus \lambda^{k-1}(N)),$$

*in particular*

$$C^\infty(I \times N) \simeq C^\infty(I, C^\infty(N)).$$

*Proof.* Define

$$G := \Lambda^k T^*N \oplus \Lambda^{k-1} T^*N.$$

Then $\pi_N^* G = \Lambda^k T^*N \oplus \Lambda^{k-1} T^*N$, i.e. at a point $(r, p) \in I \times N$ the fiber of $\pi_N^* G$ is $\Lambda^k T_p^* N \oplus \Lambda^{k-1} T_p^* N$ for every $r \in I$. Note also that

$$\begin{aligned}
\Lambda^k(T^*_{(r,p)}(I \times N)) &= \Lambda^k(T_r^* I \oplus T_p^* N) \\
&= (\Lambda^0 T_r^* I \otimes \Lambda^k T_p^* N) \oplus (\Lambda^1 T_r^* I \otimes \Lambda^{k-1} T_p^* N) \\
&= \Lambda^k T_p^* N \oplus \Lambda^{k-1} T_p^* N.
\end{aligned}$$

Hence $\Lambda^k T^*(I \times N) = \pi_N^* G$.

By Lemma 2.3, we obtain

$$C^\infty(I \times N, \Lambda^k T^*(I \times N)) \simeq C^\infty(I, C^\infty(N, \Lambda^k T^*N \oplus \Lambda^{k-1} T^*N)).$$

□

Define by $L^2(I \times N)$ the Hilbert space of the square-integrable functions on $I \times N$ with the inner product

$$\langle \varphi, \psi \rangle_{L^2(I \times N)} = \int_I \int_N \varphi \psi r^{\dim N} \, \mathrm{dvol}_N \, dr,$$

where $\varphi, \psi \in L^2(I \times N)$. Define by $L^2(I, L^2(N)) := \{\varphi : I \to L^2(N) \mid \varphi \text{ is square integrable }\}$ the Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{L^2(I, L^2(N))} = \int_I \int_N \varphi \psi \, \mathrm{dvol}_N \, dr,$$

where $\varphi, \psi \in L^2(I, L^2(N))$. Then there is a bijective unitary map

$$\Psi : L^2(I, L^2(N)) \to L^2(I \times N). \tag{2.3}$$

For $\varphi \in L^2(I, L^2(N))$ the map is defined by

$$\varphi \mapsto r^{-\frac{\dim N}{2}} \varphi.$$

## 2.3 Trace lemma

We prove here the Trace Lemma following [BS2, Appendix A] with some additional details. The Lemma will be used for the proofs in Section 3.

Let $H$ be a Hilbert space. Denote by $C_1(H)$ the trace class operators, i.e. the first Schatten class of operators. Denote by $||\cdot||_{\mathrm{tr}}$ and $||\cdot||_{HS}$ respectively trace norm and Hilbert-Schmidt operator norm.

**Lemma 2.5** (Trace Lemma)**.** *Let $T$ be a trace class operator on $L^2(\mathbb{R}, H)$. Then $T$ has a kernel $t(x,y)$, so that for $u(x) \in dom(T)$ we have $Tu(x) = \int_{-\infty}^{\infty} t(x,y)u(y)dy$ and*

$$h \mapsto t(\cdot, \cdot + h)$$

*is a continuous map from $\mathbb{R}$ into $L^1(\mathbb{R}, C_1(H))$. Furthermore,*

$$\int_{-\infty}^{\infty} ||t(x,x)||_{\mathrm{tr}} dx \leq ||T||_{\mathrm{tr}}$$

*and*

$$\int_{-\infty}^{\infty} \mathrm{tr}(t(x,x))dx = \mathrm{tr}\, T.$$

*Proof.* Since $T$ is trace class, there are two Hilbert-Schmidt operators $R, S$ such that $T = RS$ and

$$||T||_{\mathrm{tr}} = ||R||_{HS}||S||_{HS}.$$

Denote by $r(x,y)$ and $s(x,y)$ the integral kernels of $R$ and $S$ respectively. Then

$$Ru(x) = \int_{-\infty}^{\infty} r(x,y)u(y)dy,$$

where for almost all $(x,y)$, we have $||r(x,y)||_{HS} < \infty$ and

$$\int_{-\infty}^{\infty} ||r(x,y)||_{HS}^2 dxdy = ||R||_{HS}^2 < \infty.$$

The same is true for the operator $S$, hence $T = RS$ has a kernel

$$t(x,y) := \int_{-\infty}^{\infty} r(x,w)s(w,y)dw$$

such that

$$\int_{-\infty}^{\infty} ||t(x,x)||_{\mathrm{tr}} dx \leq ||R||_{HS}||S||_{HS} = ||T||_{\mathrm{tr}}. \tag{2.4}$$

Denote by $L_h u(x) := u(x-h)$ the shift operator. Since the family $L_h$ is strongly continuous, $SL_h$ is continuous with respect to the Hilbert-Schmidt norm. Therefore

$$\int_{-\infty}^{\infty} ||t(x,x+h) - t(x,x+h_0)||_{\mathrm{tr}} dx \le ||R||_{HS} ||SL_h - SL_{h_0}||_{HS} \xrightarrow[h\to h_0]{} 0,$$

and the map $h \mapsto t(\cdot, \cdot + h)$ is continuous.

It remains to show the last claim of the lemma $\int_{-\infty}^{\infty} \mathrm{tr}(t(x,x))dx = \mathrm{tr}\, T$. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a basis of $L^2(R) \otimes H$. Then

$$Ru(x) = \sum_{j,k=1}^{\infty} r_{jk} \int_{-\infty}^{\infty} (u(y), \varphi_k(y)) dy \varphi_j(x),$$

where $\sum_{j,k}^{\infty} r_{jk}^2 = ||R||_{HS}^2$ and the kernel of $R$ is

$$r(x,y) = \sum_{j,k=1}^{\infty} r_{jk} (\cdot, \varphi_k(y)) \varphi_j(x).$$

Denote by $R_n$ the operator with the kernel

$$r_n(x,y) = \sum_{0<j+k<n} r_{jk} (\cdot, \varphi_k(y)) \varphi_j(x).$$

Analogously define the operators $S_n$. Then $T_n := R_n S_n$ has kernel

$$t_n(x,y) = \int_{-\infty}^{\infty} r_n(x,w) s_n(w,y) dw = \sum_{\substack{0<j+k<n \\ 0<m+k<n}} r_{jk} s_{km} (\cdot, \varphi_m(y)) \varphi_j(x).$$

Since $\mathrm{tr}(\cdot, \varphi)\psi = (\psi, \varphi)$, we get

$$\begin{aligned} \mathrm{tr}\, T_n &= \sum_{\substack{0<j+k<n \\ 0<m+k<n}} r_{jk} s_{km} = \int_{-\infty}^{\infty} \sum_{\substack{0<j+k<n \\ 0<m+k<n}} r_{jk} s_{km} (\varphi_j(x), \varphi_m(x)) dx \\ &= \int_{-\infty}^{\infty} \mathrm{tr}(t_n(x,x)) dx. \end{aligned}$$

It was shown in (2.4) that

$$\int_{-\infty}^{\infty} ||t(x,x)||_{\mathrm{tr}} dx \le ||T||_{\mathrm{tr}},$$

hence we can take the limit

$$\mathrm{tr}\, T = \lim_{n\to\infty} \mathrm{tr}\, T_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathrm{tr}(t_n(x,x)) = \int_{-\infty}^{\infty} \mathrm{tr}(t(x,x)) dx.$$

□

## 2.4 Regularized integrals and the Singular Asymptotics Lemma

In this section we define the regularized integral over the interval $(0,\infty)$ for a certain class of locally integrable functions using the Mellin transform. We follow [L, Section 2.1]. First, we recall the definition of the Mellin transform and specify the class of functions with which we will work.

**Definition 1.** Let $H$ be a Hilbert space. For a function $f \in C_c^\infty((0,\infty),H)$, the *Mellin transform* is defined by

$$Mf(s) := \int_0^\infty x^{s-1} f(x)dx,$$

for $s \in \mathbb{C}$.

Let $p,q > 0$ and denote $L^1_{loc}(0,\infty) := L^1_{loc}((0,\infty))$.

**Definition 2.** Let $f \in L^1_{loc}(0,\infty)$ be a locally integrable function such that

$$\begin{aligned} f(x) &= \sum_{j=1}^{N}\sum_{k=0}^{m_j} a_{jk} x^{\alpha_j} \log^k x + x^p f_1(x) \\ &= \sum_{j=1}^{M}\sum_{k=0}^{m'_j} b_{jk} x^{\beta_j} \log^k x + x^{-q} f_2(x), \end{aligned}$$

where $f_1 \in L^1_{loc}([0,\infty))$, $f_2 \in L^1([1,\infty))$ and $\alpha_j, \beta_j \in \mathbb{C}$ with real parts $\mathrm{Re}(\alpha_j) \leq p-1$ increasing and $\mathrm{Re}(\beta_j) \geq -q-1$ decreasing as $j$ grows. Denote the class of such functions by $L_{p,q}(0,\infty) \subset L^1_{loc}(0,\infty)$.

Also denote

$$\begin{aligned} &L_{\infty,q}(0,\infty) := \cap_{p>0} L_{p,q}(0,\infty), \\ &L_{p,\infty}(0,\infty) := \cap_{q>0} L_{p,q}(0,\infty), \\ &L_{as}(0,\infty) := L_{\infty,\infty}(0,\infty) := \cap_{p>0} L_{p,\infty}(0,\infty). \end{aligned}$$

*Remark.* In Definition 2 the first equality reflects the behaviour of $f(x)$ as $x \to 0$ and the second equality reflects the behaviour of $f(x)$ as $x \to \infty$.

*Remark.* For $f \in L_{p,q}(0,\infty)$ and $\mathrm{Re}(s) > -\min_{1\leq j\leq N}\{\mathrm{Re}(\alpha_j)\}$, the function $x^{s-1}f(x)$ is locally integrable with respect to $x \in [0,\infty)$.

We extend the Mellin transform to $f \in L_{p,q}(0,\infty)$ by splitting it into two integrals. For $c > 0$, denote

$$(Mf)(s) := (M_{[0,c]}f)(s) + (M_{[c,\infty]}f)(s) := \int_0^c x^{s-1} f(x)dx + \int_c^\infty x^{s-1} f(x)dx.$$

The next proposition shows that the Mellin transform is well defined.

**Proposition 2.6.** *Let $p, q > 0$, $f \in L_{pq}(0,\infty)$ and $s \in \mathbb{C}$, such that $1 - p < \mathrm{Re}(s) < 1 + q$. Then*

$$(Mf)(s) = (M_{[0,c]}f)(s) + (M_{[c,\infty]}f)(s)$$

*is a meromorphic function in a strip $1-p < \mathrm{Re}(s) < 1+q$ and is independent of $c$. Moreover, the continuation of $(Mf)(s)$ may have poles at most of order $m_j + 1$ at $s = -\alpha_j$ and $m'_j + 1$ at $s = -\beta_j$ in the notations of Definition 2.*

*Proof.* Using the notations of Definition 2, we obtain

$$(M_{[0,c]}f)(s) = \sum_{j=1}^{N} \sum_{k=0}^{m_j} a_{jk} \int_0^c x^{\alpha_j+s-1} \log^k x dx + \int_0^c x^{p+s-1} f_1(x)dx.$$

We compute the integral under the sum applying integration by parts two times

$$\begin{aligned}
I_j :=& \int_0^c x^{\alpha_j+s-1} \log^k x dx \\
=& (\alpha_j+s)^{-1} x^{\alpha_j+s} \log^k x|_0^c - \int_0^c (\alpha_j+s)^{-1} x^{\alpha_j+s-1} k \log^{k-1} x dx \\
=& (\alpha_j+s)^{-1} x^{\alpha_j+s} \log^k x|_0^c - (\alpha_j+s)^{-2} x^{\alpha_j+s} k \log^{k-1} x|_0^c \\
&+ \int_0^c (\alpha_j+s)^{-2} x^{\alpha_j+s-1} k(k-1) \log^{k-2} x dx.
\end{aligned}$$

Since $\mathrm{Re}(s) + \mathrm{Re}(\alpha_j) > 0$ for every $j$, evaluation of the first and the second summands at 0 gives zero. Applying integration by parts $k$ times, we obtain

$$I_j = \sum_{i=0}^{k} \frac{(-1)^{k-i} k!}{i!} c^{\alpha_j+s} \log^i c(\alpha_j+s)^{-k-1+i},$$

hence

$$
\begin{aligned}
(M_{[0,c]}f)(s) = &\sum_{j=1}^{N}\sum_{k=0}^{m_j} a_{jk}\sum_{i=0}^{k}\frac{(-1)^{k-i}k!}{i!}c^{\alpha_j+s}\log^i c(\alpha_j+s)^{-k-1+i}\\
&+\int_0^c x^{p+s-1}f_1(x)dx.
\end{aligned}
$$

Therefore the function $(M_{[0,c]}f)(s)$ has meromorphic continuation to the half plane $\mathrm{Re}(s) > 1-p$ with poles of order $m_j+1$ at points $-\alpha_j$.

Now consider

$$
(M_{[c,\infty]}f)(s) = \sum_{j=1}^{M}\sum_{k=0}^{m_j'} b_{jk}\int_c^\infty x^{\beta_j+s-1}\log^k x dx + \int_c^\infty x^{-q+s-1}f_2(x)dx.
$$

Compute the integral under the sum

$$
\begin{aligned}
I_j' :=& \int_c^\infty x^{\beta_j+s-1}\log^k x dx\\
=&(\beta_j+s)^{-1}x^{\beta_j+s}\log^k x|_c^\infty - \int_c^\infty (\beta_j+s)^{-1}x^{\beta_j+s-1}k\log^{k-1} x dx\\
=&(\beta_j+s)^{-1}x^{\beta_j+s}\log^k x|_c^\infty - (\beta_j+s)^{-2}x^{\beta_j+s}k\log^{k-1} x|_c^\infty\\
&+\int_c^\infty (\beta_j+s)^{-2}x^{\beta_j+s-1}k(k-1)\log^{k-2} x dx.
\end{aligned}
$$

Since $\mathrm{Re}(s) < 1+q \le -\mathrm{Re}(\beta_j)$, we have $\mathrm{Re}(s)+\mathrm{Re}(\beta_j) < 0$ for every $j$, and evaluation of the first and the second term at $\infty$ both give zero. Apply integration by parts $k$ times to obtain

$$
I_j' = -\sum_{i=0}^{k}\frac{(-1)^{k-i}k!}{i!}c^{\beta_j+s}\log^i c(\beta_j+s)^{-k-1+i},
$$

thus

$$
\begin{aligned}
(M_{[c,\infty]}f)(s) = &-\sum_{j=1}^{M}\sum_{k=0}^{m_j'} b_{jk}\sum_{i=0}^{k}\frac{(-1)^{k-i}k!}{i!}c^{\beta_j+s}\log^i c(\beta_j+s)^{-k-1+i}\\
&+\int_c^\infty x^{-q+s-1}f_2(x)dx.
\end{aligned}
$$

Therefore the function $(M_{[c,\infty]}f)(s)$ has meromorphic continuation to the half plane $\mathrm{Re}(s) < 1+q$ with poles of order $m_j'+1$ at points $-\beta_j$.

By definition,

$$0 = f(c) - f(c) = \sum_{j=1}^{N}\sum_{k=0}^{m_j} a_{jk}c^{\alpha_j}\log^k c + c^p f_1(c)$$
$$- \sum_{j=1}^{M}\sum_{k=0}^{m_j'} b_{jk}c^{\beta_j}\log^k c - c^{-q} f_2(c).$$

Hence $(Mf)(s) = (M_{[0,c]}f)(s) + (M_{[c,\infty]}f)(s)$ does not depend on the choice of $c > 0$. However, the poles may cancel, therefore they may be of lower order. $\square$

Let $f$ be a meromorphic function. Denote by $\mathrm{Res}_k\, f(z_0)$ the coefficient of $(z-z_0)^{-k}$ in the Laurent expansion of $f$ near $z_0$

$$f(z) = \sum_{k=-m}^{\infty} \mathrm{Res}_{-k}\, f(z_0)(z-z_0)^k.$$

**Definition 3.** Let $f \in L_{p,q}(0,\infty)$. A *regularized integral* is the constant coefficient in the Laurent expansion near $s=1$ of the Mellin transform of $f$, i.e.

$$⨍_0^\infty f(x)dx := \mathrm{Res}_0(Mf)(1).$$

Now we are ready to state the Singular Asymptotics Lemma.

Let $C := \{|\arg\zeta| < \pi - \epsilon\}$ for some $\epsilon > 0$.

**Lemma 2.7** (Singular Asymptotics Lemma, [BS2, p.372])**.** *Let $\sigma(r,\zeta)$ be defined on $\mathbb{R}\times C$ and satisfy the following conditions*

*(1) $\sigma(r,\zeta)$ is $C^\infty$ with respect to $r$ and has analytic derivatives with respect to $\zeta$;*

*(2) there exist Schwartz functions $\sigma_{\alpha j}(r)\in\mathscr{S}(\mathbb{R})$ such that for $|\zeta|\geq 1$ and $0\leq r\leq|\zeta|/C_0$,*

$$\left|r^J\partial_r^K\left(\sigma(r,\zeta)-\sum_{Re\alpha\geq -M}\sum_{j=0}^{J_\alpha}\sigma_{\alpha j}(r)\zeta^\alpha\log^j\zeta\right)\right|\leq C_{JKM}|\zeta|^{-M};$$

*(3) (integrability condition) the derivatives $\sigma^{(j)}(r,\zeta):=\partial_r^j\sigma(r,\zeta)$ satisfy uniformly for $0\leq t\leq 1$ and $|\xi|=C_0$*

$$\int_0^1\int_0^1 s^j|\sigma^{(j)}(st,s\xi)|dsdt\leq C_j.$$

*Then*

$$\begin{aligned}\int_0^\infty\sigma(r,rz)dr\sim_{z\to\infty}&\sum_{k=0}^\infty z^{-k-1}⨍_0^\infty\frac{\zeta^k}{k!}\sigma^{(k)}(0,\zeta)d\zeta\\&+\sum_\alpha\sum_{j=0}^{J_\alpha}⨍_0^\infty\sigma_{\alpha j}(r)(rz)^\alpha\log^j(rz)dr\\&+\sum_{\alpha=-1}^{-\infty}\sum_{j=0}^{J_\alpha}\sigma_{\alpha j}^{(-\alpha-1)}(0)\frac{z^\alpha\log^{j+1}z}{(j+1)(-\alpha-1)!}.\end{aligned}$$

*Remark.* Above $\alpha$ is any sequence of complex numbers with $\mathrm{Re}(\alpha)\to-\infty$. The last sum in the expansion includes only those $\alpha$ that happen to be negative integers. $J_\alpha$ is the biggest power of $\log\zeta$ that occurs for $\alpha$.

# 3 Main computations and methods

The aim of this chapter is to prove Theorem 1.1. First, we recall the asymptotic expansion of the heat kernel along the diagonal, which is a local result and does not require completeness of the manifold. The expansion is given in terms of the curvature tensor and its covariant derivatives. In the case of a compact manifold, one integrates the terms in the local expansion over the manifold and obtains the classical heat kernel expansion (1.2). In the case of a non-complete manifold $(M, g)$ with conic singularities, defined in Section 2.1, we compute the curvature tensor near the conic point and observe that the integrals over the manifold in general diverge near the conic point. Then we use the Singular Asymptotics Lemma to obtain the heat trace expansion from the local heat kernel expansion.

## 3.1 Local expansion of the heat kernel

Let $(M, g)$ be a Riemannian manifold, possibly non-complete, and $\Delta$ be the Laplace-Beltrami operator on $(M, g)$. For $(p, q) \in M \times M$ denote the heat kernel by $e^{-t\Delta}(p, q)$. The heat kernel along the diagonal $(p, p) \in M \times M$ is denoted by $e^{-t\Delta}(p)$. The next proposition gives an expansion of the heat kernel along the diagonal on any compact subset of $(M, g)$.

**Theorem 3.1** ( [BGM, Section III.E] ). *Let $K \subset M$ be any compact set and $p \in K$. There is an asymptotic expansion of the heat kernel along the diagonal*

$$\|e^{-t\Delta}(p) - (4\pi t)^{-\frac{\dim M}{2}} \sum_{i=0}^{j} t^i u_i(p)\| \leq C_j(K) t^{j+1},$$

*where $C_j(K)$ is some constant which depends on the compact set $K$. Moreover, $u_0(p) \equiv 1$ and $u_1(p) = \frac{1}{6} \operatorname{Scal}(p)$, where $\operatorname{Scal}(p)$ is the scalar curvature at $p \in M$, and all $u_i(p)$ are polynomials on the curvature tensor and its covariant derivatives.*

Furthermore [G2, p.201 Theorem 3.3.1]

$$u_2(p) = \frac{1}{360} \left(12\Delta \operatorname{Scal}(p) + 5 \operatorname{Scal}(p)^2 - 2|\operatorname{Ric}(p)|^2 + 2|\mathrm{R}(p)|^2\right), \qquad (3.1)$$

where

$$|\mathrm{R}(p)|^2 := \mathrm{R}_{ijkl}(p)\, \mathrm{R}_{ijkl}(p) g^{ii}(p) g^{jj}(p) g^{kk}(p) g^{ll}(p),$$

$$|\operatorname{Ric}(p)|^2 := \operatorname{Ric}_{ij}(p) \operatorname{Ric}_{ij}(p) g^{ii}(p) g^{jj}(p).$$

Above, $\mathrm{R}_{ijkl}(p)$ is the Riemann curvature tensor, $\mathrm{Ric}_{ij}(p)$ is the Ricci tensor, $\mathrm{Scal}(p)$ is the scalar curvature.

The heat operator is closely related to the resolvent operator by the Cauchy's differentiation formula. For a positively oriented closed path $\gamma$ in the complex plane surrounding the spectrum of $\Delta$ and for $d \in \mathbb{N}$, we have

$$e^{-t\Delta} = -t^{1-d}\frac{(d-1)!}{2\pi i}\int_\gamma e^{-t\mu}(\Delta-\mu)^{-d}d\mu. \tag{3.2}$$

To interpolate between the expansion of the heat trace and the expansion of the resolvent trace we will use the following formulas

$$\begin{aligned}\int_\gamma e^{-t\mu}(-\mu)^{-n}d\mu &= 2\pi i\,\mathrm{Res}_{\mu=0}(e^{-t\mu}(-\mu)^{-n})\\ &= \frac{2\pi i}{(n-1)!}\lim_{\mu\to 0}\left(\frac{d}{d\mu}\right)^{n-1}(-1)^n e^{-t\mu} = -\frac{2\pi i}{\Gamma(n)}t^{n-1}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_\gamma e^{-t\mu}(-\mu)^{-n}\log(-\mu)d\mu = -\frac{d}{dn}\int_\gamma e^{-t\mu}(-\mu)^{-n}d\mu &= \frac{d}{dn}\left(\frac{2\pi i}{\Gamma(n)}t^{n-1}\right)\\ &= 2\pi i\frac{t^{n-1}\log t\Gamma(n) - t^{n-1}\Gamma(n)'}{\Gamma(n)^2}\\ &= \frac{2\pi i}{\Gamma(n)}t^{n-1}\log t - \frac{2\pi i}{\Gamma(n)}\frac{\Gamma'(n)}{\Gamma(n)}t^{n-1}.\end{aligned} \tag{3.4}$$

On $(M,g)$ by Theorem 3.1, we have the local asymptotic expansion of the heat kernel along the diagonal. Then using Cauchy's differentiation formula (3.2), (3.3) and (3.4), we obtain the expansion of the kernel of the resolvent along the diagonal for $p \in K \subset M$. Denote $z^2 := -\mu$. By [G, p.61, Lemma 1.7.2],

$$\left\|(\Delta+z^2)^{-d}(p) - (4\pi)^{-\frac{m}{2}}\sum_{j=d-\frac{m}{2}}^{k} z^{-2j}u_{j+\frac{m}{2}-d}(p)\frac{\Gamma(j)}{(d-1)!}\right\| \le \tilde{C}_k(K)z^{-2j-2},$$

for some $\tilde{C}(K) > 0$. For convenience denote $l := j + \frac{m}{2} - d$, then

$$\left\| (\Delta + z^2)^{-d}(p) - (4\pi)^{-\frac{m}{2}} \sum_{l=0}^{k+m/2-d} z^{-2d+m-2l} u_l(p) \frac{\Gamma(-\frac{m}{2}+d+l)}{(d-1)!} \right\| \le \tilde{C}_k(K) z^{-2d+m-2l-2}, \tag{3.5}$$

where $u_0(p) \equiv 1$ and $u_1(p) = \frac{\mathrm{Scal}(p)}{6}$.

## 3.2 Curvature tensor in polar coordinates

In this section we give explicit formulas for the curvature tensors in the neighbourhood $(U, g_{\text{conic}})$ of the conic singularity in terms of the curvature tensors on the cross-section manifold $(N, g_N)$ of dimension $n$. Let $x = (x^1, \ldots, x^n)$ be local coordinates on $(N, g_N)$ and $p = (r, x^1, \ldots, x^n) \in U$. For $i, j \in \{0, 1, \ldots, n\}$ denote by $\tilde{g}_{ij}$ the components of the metric tensor $g_{\text{conic}} = dr^2 + r^2 g_N$, and by $g_{ij}$ for $i, j \in \{1, \ldots, n\}$ the components of the metric tensor $g_N$. Then

$$\tilde{g}_{00} = 1, \quad \tilde{g}_{i0} = \tilde{g}_{0i} = 0, \text{ for } i > 0,$$

and

$$\tilde{g}_{ij} = r^2 g_{ij}, \text{ for } i, j > 0.$$

We use the standard notations for the tensors that correspond to the metric $g_{ij}$. For tensors corresponding to the metric $\tilde{g}_{ij}$, we use the same notations, but with tildes. To stress that the tensor depends on a point we use $\tilde{g}_{ij}(p) = \tilde{g}_{ij}(r, x)$. If it is clear, we may omit a point $p$ to simplify the notations. Denote the derivative of $\tilde{g}_{ij}$ with respect to the $k$-th coordinate by $\tilde{g}_{ij,k}$. If $i$ or $j$ or both are equal to zero then $\tilde{g}_{ij,k} = 0$ for any $k \in 0, 1, \ldots, n$. Suppose $i, j \neq 0$, then

$$\tilde{g}_{ij,k}(r, x) = \begin{cases} 2r g_{ij}(x), & \text{if } k = 0, \\ r^2 g_{ij,k}(x), & \text{if } k \neq 0. \end{cases}$$

The Christoffel symbols are of course

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} \tilde{g}^{im} \left( \tilde{g}_{mj,k} + \tilde{g}_{mk,j} - \tilde{g}_{jk,m} \right),$$

but now we express them in terms of the Christoffel symbols $\Gamma^i_{jk}$ and the metric tensor $g_{ij}$.

Let $i = 0$

$$\tilde{\Gamma}^0_{jk} = \begin{cases} 0, \text{ if } j = 0 \text{ or } k = 0, \\ -rg_{jk}, \text{ otherwise }. \end{cases}$$

Assume $i \neq 0$ and let $j = 0$, then

$$\begin{aligned}
\tilde{\Gamma}^i_{0k} = \tilde{\Gamma}^i_{k0} &= \frac{1}{2}\tilde{g}^{im}\left(\tilde{g}_{m0,k} + \tilde{g}_{mk,0} - \tilde{g}_{0k,m}\right) \\
&= \frac{1}{2}r^{-2}g^{im}\left(\tilde{g}_{mk,0}\right) \\
&= \frac{1}{2}r^{-2}g^{im}\left(2rg_{mk}\right) \\
&= r^{-1}\delta^i_k.
\end{aligned}$$

If both $j = k = 0$, then $\tilde{\Gamma}^i_{00} = 0$. Assume that $i, j$ and $k$ are all non-zero. Then

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}.$$

The scalar curvature $\tilde{\mathrm{Scal}}$ can be expressed in terms of the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ in the following way

$$\begin{aligned}
\tilde{\mathrm{Scal}} =& \tilde{g}^{ij}\left(\tilde{\Gamma}^m_{ij,m} - \tilde{\Gamma}^m_{im,j} + \tilde{\Gamma}^l_{ij}\tilde{\Gamma}^m_{ml} - \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jl}\right) \\
=& \tilde{g}^{0j}\left(\tilde{\Gamma}^m_{0j,m} - \tilde{\Gamma}^m_{0m,j} + \tilde{\Gamma}^l_{0j}\tilde{\Gamma}^m_{ml} - \tilde{\Gamma}^l_{0m}\tilde{\Gamma}^m_{jl}\right) \\
&+ \sum_{i\neq 0}\tilde{g}^{ij}\left(\tilde{\Gamma}^m_{ij,m} - \tilde{\Gamma}^m_{im,j} + \tilde{\Gamma}^l_{ij}\tilde{\Gamma}^m_{ml} - \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jl}\right) \\
=:& \tilde{\mathrm{Scal}}_0 + \tilde{\mathrm{Scal}}_1.
\end{aligned}$$

Now we compute $\tilde{\mathrm{Scal}}_0$ and $\tilde{\mathrm{Scal}}_1$.

Since $\tilde{g}^{0j}$ is equal to zero for any $j$, $j \neq 0$, we have

$$\begin{aligned}
\tilde{\mathrm{Scal}}_0 =& \left(\tilde{\Gamma}^m_{00,m} - \tilde{\Gamma}^m_{0m,0} + \tilde{\Gamma}^l_{00}\tilde{\Gamma}^m_{ml} - \tilde{\Gamma}^l_{0m}\tilde{\Gamma}^m_{0l}\right) \\
=& -\partial_m(r^{-1}\delta^m_m) - r^{-1}\delta^l_m r^{-1}\delta^m_l = r^{-2}n - r^{-2}n^2 = -r^{-2}n(n-1)
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{\text{Scal}}_1 =& \sum_{i\neq 0} \tilde{g}^{ij} \left( \tilde{\Gamma}^m_{ij,m} - \tilde{\Gamma}^m_{im,j} + \tilde{\Gamma}^l_{ij}\tilde{\Gamma}^m_{ml} - \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jl} \right) \\
=& r^{-2} g^{ij} \Bigg( -g_{ij} + \sum_{m\neq 0} [\tilde{\Gamma}^m_{ij,m} - \tilde{\Gamma}^m_{im,j}] - g_{ij}\delta^m_m + \sum_{l\neq 0} \tilde{\Gamma}^l_{ij}\tilde{\Gamma}^m_{ml} + g_{im}\delta^m_j \\
& - \sum_{l\neq 0} \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jl} + \delta^l_i g_{jl} \Bigg) \\
=& r^{-2} g^{ij} \left( -g_{ij} + [\Gamma^m_{ij,m} - \Gamma^m_{im,j}] - g_{ij} n + \Gamma^l_{ij}\Gamma^m_{ml} + g_{ij} n - \Gamma^l_{im}\Gamma^m_{jl} + g_{ij} \right) \\
=& r^{-2} g^{ij} \left( \Gamma^m_{ij,m} - \Gamma^m_{im,j} + \Gamma^l_{ij}\Gamma^m_{ml} - \Gamma^l_{im}\Gamma^m_{jl} \right) = r^{-2}\, \text{Scal},
\end{aligned}
$$

where Scal is the scalar curvature at $x \in N$.

Therefore the scalar curvature in the polar coordinates $p = (r, x^1, \ldots, x^n)$ is

$$
\tilde{\text{Scal}}(p) = r^{-2}\big( \text{Scal}(x) - n(n-1) \big), \tag{3.6}
$$

where $p = (r, x) \in U$ and $x \in N$ and $\tilde{\text{Scal}}(p)$ is the scalar curvature on $(U, g_{\text{conic}})$ and $\text{Scal}(x)$ is the scalar curvature on $(N, g_N)$.

Recall that the Riemann curvature tensor and the Ricci tensor can be written using the Christoffel symbols as follows

$$
\text{R}^i_{jkl} = \partial_k \Gamma^i_{jl} + \Gamma^i_{kp}\Gamma^p_{jl} - \partial_l \Gamma^i_{jk} - \Gamma^i_{lp}\Gamma^p_{jk}
$$

and

$$
\text{Ric}_{ij} = \partial_m \Gamma^m_{ij} + \Gamma^m_{mp}\Gamma^p_{ij} - \partial_j \Gamma^m_{mi} - \Gamma^m_{jp}\Gamma^p_{im}.
$$

Now we express the Riemann tensor $\tilde{\text{R}}^i_{jkl}$ in terms of the Riemann curvature tensor $\text{R}^i_{jkl}$.

Let $i = 0$ and $i, j, k$ be nonzero, then

$$
\tilde{\text{R}}^0_{jkl} = 0,
$$

also

$$
\tilde{\text{R}}^i_{0kl} = \tilde{\text{R}}^i_{j0l} = \tilde{\text{R}}^i_{j00} = \tilde{\text{R}}^i_{000} = 0
$$

and

$$
\tilde{\text{R}}^i_{00l} = \tilde{\text{R}}^i_{0l0} = -r^{-2}\delta^i_l.
$$

If none of the indices $i, j, k, l$ is zero, we obtain

$$
\tilde{\text{R}}_{ijkl}(p) = r^{-2}\big( \text{R}_{ijkl}(x) - g_{ip}(x) g_{jm}(x) (\delta^p_k \delta^m_l - \delta^p_l \delta^m_k) \big). \tag{3.7}
$$

Similarly, for the tensor Ricci

$$
\tilde{\text{Ric}}_{ij}(p) = r^{-2} \left( \text{Ric}_{ij}(x) - (n-1) g_{ij}(x) \right). \tag{3.8}
$$

## 3.3 The Laplace operator on the infinite cone

In this section we obtain an expression (3.11) of the Laplace-Beltrami operator on an infinite cone, and show that it satisfies the scaling property.

The Laplace-Beltrami operator $\Delta := d^\dagger d$ acting on $C_c^\infty((0,\varepsilon)\times N)$ in the Hilbert space $L^2((0,\infty)\times N)$ with measure $r^n\,\mathrm{dvol}_N\,dr$ can be written in terms of partial derivatives with respect to the local coordinates. We have $d^\dagger = -*d*$, where $*$ is the Hodge-star operator on $(U, g_{\text{conic}})$.

Let $p = (r, x^1, \dots, x^n) \in U$ and $f(r, x^1 \dots, x^n) \in C_c^\infty(U)$. Denote $g := \det g_N = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Some computations that will be used later are given in the next proposition. Choose coordinates on $(N, g_N)$ such that the metric is diagonal.

**Proposition 3.2.** *Let $A \in C_c^\infty(U)$. Then*

*a)* $*(Adr) = Ar^n g^{1/2} dx^1 \wedge \dots \wedge dx^n$;

*b)* $*(Adx^i) = A(-1)^i r^{n-2} g^{ii} g^{1/2} dr \wedge dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n$;

*c)* $*(Adr \wedge dx^1 \wedge \dots \wedge dx^n) = Ar^{-n} g^{-1/2}$.

*Proof.* Let $X \in C_c^\infty(U)$.

a) Let $X$ be such that $*(Adr) = X dx^1 \wedge \dots \wedge dx^n$, then by the definition of the Hodge star operator, for any one-form $\alpha dr$ we have

$$\alpha dr \wedge X dx^1 \wedge \dots \wedge dx^n = \alpha A r^n g^{1/2} dr \wedge dx^1 \wedge \dots \wedge dx^n,$$

hence $X = Ar^n g^{1/2}$.

b) Let $X$ be such that $*(Adx^i) = X dr \wedge dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n$. Then for any one-form $\alpha dx^i$

$$\alpha dx^i \wedge X dr \wedge dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n = \alpha A r^{-2} g^{ii} r^n g^{1/2} dr \wedge dx^1 \wedge \dots \wedge dx^n,$$

hence

$$(-1)^i \alpha X = \alpha A g^{ii} r^{n-2} g^{1/2}.$$

Consequently, $X = (-1)^i A g^{ii} r^{n-2} g^{1/2}$.

c) Let $X$ be such that $*(Adr \wedge dx^1 \wedge \dots \wedge dx^n) = X$, then for any $n+1$-form $\alpha dr \wedge dx^1 \wedge \dots \wedge dx^n$

$$\alpha dr \wedge dx^1 \wedge \dots \wedge dx^n X = \alpha A r^{-2n} g^{-1} r^n g^{1/2} dr \wedge dx^1 \wedge \dots \wedge dx^n.$$

Hence $X = Ar^{-n} g^{-1/2}$. □

Let $f(r, x^1, \dots, x^n)$ be a smooth function with compact support in $(U, g_{\text{conic}})$ and apply the operator $\Delta = - * d * d$ to it. By Proposition 3.2 we obtain

$$
\begin{aligned}
f(r, x^1, \dots, x^n) \overset{d}{\mapsto} & \partial_r f dr + \sum_{i=1}^{n} \partial_{x^i} f dx^i \\
\overset{*}{\mapsto} & r^n g^{1/2} \partial_r f dx^1 \wedge \dots \wedge dx^n \\
& + \sum_{i=1}^{n} (-1)^i r^{n-2} g^{ii} g^{1/2} \partial_{x^i} f dr \wedge dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n \\
\overset{d}{\mapsto} & dr \wedge dx^1 \wedge \dots \wedge dx^n \Bigg( r^n g^{1/2} \partial_r^2 f + n r^{n-1} g^{1/2} \partial_r f \\
& + \sum_{i=1}^{n} \big( r^{n-2} g^{ii} g^{1/2} \partial_{x^i}^2 f + r^{n-2} g^{1/2} \partial_{x^i} f \partial_{x^i} g^{ii} \\
& + \frac{1}{2} r^{n-2} g^{-1/2} g^{ii} \partial_{x^i} f \partial_{x^i} g \big) \Bigg) \\
\overset{*}{\mapsto} & \partial_r^2 f + n r^{-1} \partial_r f \\
& + \sum_{i=1}^{n} \big( r^{-2} g^{ii} \partial_{x^i}^2 f + r^{-2} \partial_{x^i} f \partial_{x^i} g^{ii} + \frac{1}{2} r^{-2} g^{-1} g^{ii} \partial_{x^i} f \partial_{x^i} g \big) \\
\overset{-}{\mapsto} & - \partial_r^2 f - n r^{-1} \partial_r f \\
& - r^{-2} \sum_{i=1}^{n} \big( g^{ii} \partial_{x^i}^2 f + \partial_{x^i} f \partial_{x^i} g^{ii} + \frac{1}{2} g^{-1} g^{ii} \partial_{x^i} f \partial_{x^i} g \big).
\end{aligned}
\tag{3.9}
$$

Now consider the Laplace-Beltrami operator $\Delta_N$ on $(N, g_N)$

$$
\begin{aligned}
\Delta_N f &= - *d * df = - * d * \sum_{i=1}^{n} \partial_{x^i} f dx^i \\
&= - *d \left( \sum_{i=1}^{n} (-1)^i g^{1/2} g^{ii} \partial_{x^i} f dx^1 \wedge \dots \hat{x^i} \dots \wedge dx^n \right) \\
&= - * \Bigg( g^{1/2} g^{ii} \partial_{x^i}^2 f + g^{1/2} \partial_{x^i} f \partial_{x^i} g^{ii} \\
&\quad + \frac{1}{2} g^{-1/2} g^{ii} \partial_{x^i} f \partial_{x^i} g \Bigg) dx^1 \wedge \dots \wedge dx^n \\
&= - \left( g^{ii} \partial_{x^i}^2 f + \partial_{x^i} f \partial_{x^i} g^{ii} + \frac{1}{2} g^{-1} g^{ii} \partial_{x^i} f \partial_{x^i} g \right).
\end{aligned}
\tag{3.10}
$$

By (3.9) and (3.10)

$$\Delta = -\partial_r^2 - nr^{-1}\partial_r + r^{-2}\Delta_N.$$

In (2.3), we defined a unitary map between the Hilbert spaces

$$\Psi : L^2((0,\infty), L^2(N)) \to L^2((0,\infty)\times N),$$
$$\Psi : f \mapsto r^{-n/2} f$$

for $f \in L^2((0,\infty), L^2(N))$.

Define the operator

$$T := \Psi^{-1}\Delta\Psi,$$

acting on $C_c^\infty((0,\infty), C^\infty(N))$ in the Hilbert space $L^2((0,\infty), L^2(N))$.

Then

$$\begin{aligned}
Tf =& r^{n/2}\Delta r^{-n/2} f \\
=& r^{n/2}\left(-\partial_r^2 - nr^{-1}\partial_r + r^{-2}\Delta_N\right) r^{-n/2} f \\
=& -r^{n/2}\partial_r\left(r^{-n/2}\partial_r f - \frac{n}{2} r^{-n/2-1} f\right) - nr^{n/2-1}\left(r^{-n/2}\partial_r f - \frac{n}{2} r^{-n/2-1} f\right) \\
& + r^{-2}\Delta_N f \\
=& -\partial_r^2 f - \frac{n}{2}\left(\frac{n}{2}+1\right) r^{-2} f + r^{-2}\frac{n^2}{2} f + r^{-2}\Delta_N f \\
=& -\partial_r^2 f + r^{-2}\left(\frac{n}{2}\left(\frac{n}{2}-1\right) f + \Delta_N f\right).
\end{aligned}$$

Finally

$$T = -\partial_r^2 + r^{-2}\left(\frac{n}{2}\left(\frac{n}{2}-1\right) + \Delta_N\right). \tag{3.11}$$

Note that since

$$\frac{n}{2}(\frac{n}{2}-1) + \Delta_N = \left(\frac{n}{2}-\frac{1}{2}\right)^2 + \Delta_N - \frac{1}{4} \geq -\frac{1}{4},$$

the operator $T$ is bounded below. It is also symmetric. The Friedrichs extension of $T$ satisfies the scaling property, see Lemma 3.5. Below we deal only with this extension so to simplify the notation denote it by $T$.

## 3.4 The resolvent trace expansion

Now we will consider resolvent of operator $T$. Since the manifold $(N, g_N)$ is compact, the Laplace-Beltrami operator $\Delta_N$ has discrete spectrum. There is a basis of the Hilbert space $L^2(N)$ that consists of the corresponding eigenfunctions. Let $\lambda \in \operatorname{spec} \Delta_N$ be eigenvalues of $\Delta_N$. Then

$$(T+z^2)^{-1} = \otimes_\lambda \left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2}-1\right)\right) + z^2\right)^{-1} \otimes_\lambda \pi_\lambda,$$

where $\pi_\lambda$ is the projection on the $\lambda$-eigenspace of $\Delta_N$.

The proof of the next lemma may appear similar to [BS3, Lemma 4.1], but here our proof necessitates non-trivial subtle differences.

**Lemma 3.3.** *Let $\nu = \sqrt{\lambda + (n-1)^2/4}$, Im $z^2 \neq 0$ and $0 < r_1 \leq r_2 < \infty$, then the resolvent*

$$\left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2}-1\right)\right) + z^2\right)^{-1}$$

*is an integral operator with kernel given by*

$$\left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2}-1\right)\right) + z^2\right)^{-1}(r_1, r_2) = (r_1 r_2)^{1/2} I_\nu(r_1 z) K_\nu(r_2 z),$$

*where $I_\nu(r_1 z)$ and $K_\nu(r_2 z)$ are the modified Bessel functions of the first and second type respectively.*

*Proof.* Let $v_1(r,z)$ and $v_2(r,z)$ be two linearly independent non-zero solutions of

$$\left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2}-1\right)\right) + z^2\right) u(r) = 0, \tag{3.12}$$

then by [DS, Theorem XIII.3.16], the resolvent is an integral operator and the kernel of the resolvent is

$$\begin{aligned} &\left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2}-1\right)\right) + z^2\right)^{-1}(r_1, r_2) \\ &\quad = (v_1' v_2 - v_1 v_2')^{-1}(r_1, z) v_1(r_1, z) v_2(r_2, z), \end{aligned} \tag{3.13}$$

for $0 < r_1 < r_2 < \infty$.

We now find $v_1$ and $v_2$ by solving (3.12). Put

$$u(r) =: r^{1/2} w(r),$$

then

$$\partial_r u(r) = \frac{1}{2} r^{-1/2} w(r) + r^{1/2} \partial_r w(r)$$

and

$$\partial_r^2 u(r) = -\frac{1}{4} r^{-3/2} w(r) + r^{-1/2} \partial_r w(r) + r^{1/2} \partial_r^2 w(r).$$

Hence (3.12) becomes the modified Bessel equation

$$\left(r^2\partial_r^2 + r\partial_r - \left(r^2z^2 + \lambda + \frac{(n-1)^2}{4}\right)\right) w(r) = 0.$$

Since $\nu = \sqrt{\lambda + (n-1)^2/4}$, the general solution is generated by the modified Bessel functions

$$w(r) = C_1 I_\nu(rz) + C_2 K_\nu(rz),$$

where $C_1, C_2 \in \mathbb{R}$. Substitute $w(r) = r^{-1/2}u(r)$ to obtain

$$u(r) = C_1 r^{1/2} I_\nu(rz) + C_2 r^{1/2} K_\nu(rz).$$

Note that the modified Bessel function of the first kind, $I_\nu(r)$ with $\nu > 0$, grows exponentially as $r \to \infty$. It tends to zero as $r \to 0$. On the other hand the modified Bessel function of the second kind $K_\nu(r)$ tends to zero as $r \to \infty$, and grows as $r \to 0$. Using the boundary condition at $r = 0$ and the boundary condition at infinity $\lim_{r\to\infty} u(r) = 0$, we derive two linearly independent solutions of (3.12)

$$v_1(r) = r^{1/2} I_\nu(rz)$$

and

$$v_2(r) = r^{1/2} K_\nu(rz).$$

To find the kernel of the resolvent we use the formula (3.13). Since $I'_\nu(x)K_\nu(x) - I_\nu(x)K'_\nu(x) = x^{-1}$,

$$\begin{aligned} &(v_1'v_2 - v_1v_2')^{-1}(r_1, z) \\ =&\left(\frac{1}{2}r_1^{-1/2}I'_\nu(r_1z) + zr_1^{1/2}I'_\nu(r_1z)\right) r^{1/2}K_\nu(r_1z) \\ &- r_1^{1/2}I_\nu(r_1z)\left(\frac{1}{2}r_1^{-1/2}K'_\nu(r_1z) + zr_1^{1/2}K'_\nu(r_1z)\right) \\ =&r_1z\big(I'_\nu(r_1z)K_\nu(r_1z) - I_\nu(r_1z)K'_\nu(r_1z)\big) = 1. \end{aligned} \tag{3.14}$$

By (3.13)

$$\begin{aligned} &\left(-\partial_r^2 + r^{-2}\left(\lambda + \frac{n}{2}\left(\frac{n}{2} - 1\right)\right) + z^2\right)^{-1}(r_1, r_2) \\ =&(r_1r_2)^{1/2} I_\nu(r_1z)K_\nu(r_2z). \end{aligned} \tag{3.15}$$

□

Since $(M, g)$ is a non-complete manifold, $T^{-1}$ may be not trace class, but for some $\varphi_1, \varphi_2 \in C_c^\infty(M)$

$$\varphi_1 T^{-1} \varphi_2 \in C_p(L^2((0,\varepsilon), L^2(N))),$$

where $C_p(H)$ is the $p$-th Schatten class of operators. It follows from the resolvent identity [W, Theorem 5.13] and the Hölder inequality for Schatten norms, that for $d > p$ and $\varphi \in C_c^\infty(M)$

$$\varphi(T + z^2)^{-d} \in C_1(L^2(M)),$$

with uniform trace norm estimate in

$$\{z \in \mathbb{C} \mid |arg z| < \delta\}, \qquad 0 < \delta < \pi/2.$$

It is shown in [BS2, pp. 400-409] that for any function $\varphi \in C_c^\infty(\mathbb{R})$ the operator $\varphi(r)(T + z^2)^{-d}$ is trace class for $d > \dim M/2 = m/2$.

Let $\varphi$ be a smooth function on $(M, g)$ with support in the neighbourhood $(U, g_{\text{conic}})$ of the singularity, such that it depends only on the radial coordinate $r$. For each fixed $r \in (0, \varepsilon)$ and $p = (r, x^1, \dots, x^n) \in M$ by (3.5), we have the expansion of the heat kernel along the diagonal

$$\begin{aligned} &\varphi(r)(T + z^2)^{-d}(p) \sim_{|z|\to\infty} \\ &(4\pi)^{-\frac{m}{2}} \varphi(r) \sum_{j=0}^{\infty} z^{-2d+m-2j} u_j(p) \frac{\Gamma(-\frac{m}{2} + d + j)}{(d-1)!}. \end{aligned} \tag{3.16}$$

**Proposition 3.4.** *Let $\varphi(r)$ be a smooth function with compact support near $r = 0$ and with $\varphi(r) \equiv 1$ in a small neighbourhood of $r = 0$. Let $d > m/2$ and $z \in \mathbb{C}$ such that $|arg(z)| < \frac{\pi}{2}$, then*

$$\begin{aligned} &\mathrm{tr}(\varphi(r)(T + z^2)^{-d}) \\ &= \int_0^\infty tr_{L^2(N)}(\varphi(r)(T + z^2)^{-d}) dr \\ &= \int_0^\infty \varphi(r) \frac{r^{2d-1}}{(d-1)!} \left( -\frac{1}{2rz} \frac{\partial}{\partial(rz)} \right)^{d-1} \sum_\nu I_\nu(rz) K_\nu(rz) dr \\ &= \int_0^\infty \varphi(r) \sigma(r, rz) dr. \end{aligned}$$

*Above we sum over all $\nu = \sqrt{\lambda + (n-1)^2/4}$ such that $\lambda \in \operatorname{spec} \Delta_N$, and*

$$\sigma(r, rz) := \frac{r^{2d-1}}{(d-1)!} \left( -\frac{1}{2rz} \frac{\partial}{\partial(rz)} \right)^{d-1} \sum_\nu I_\nu(rz) K_\nu(rz).$$

*Proof.* The first equality follows from Lemma 2.5. To prove that second equality we use Lemma 3.3 and the formula

$$(T+z^2)^{-d} = \frac{1}{(d-1)!}\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{d-1}(T+z^2)^{-1}.$$

We obtain

$$\begin{aligned} tr_{L^2(N)}(T+z^2)^{-d} =& \frac{1}{(d-1)!}\left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)^{d-1} tr_{L^2(N)}(T+z^2)^{-1} \\ =& \frac{r^{2d-1}}{(d-1)!}\left(-\frac{1}{2rz}\frac{\partial}{\partial(rz)}\right)^{d-1}\sum_{\nu} I_\nu(rz)K_\nu(rz). \end{aligned}$$

□

In Proposition 3.4, we define the following function

$$\sigma(r,\zeta) = tr_{L^2(N)}(T+\zeta^2/r^2)^{-d}. \tag{3.17}$$

By (3.16), we have an asymptotic expansion

$$\begin{aligned} & tr_{L^2(N)}(T+\zeta^2/r^2)^{-d} \sim_{\zeta\to\infty} \\ & (4\pi)^{-\frac{m}{2}}\sum_{j=0}^{\infty}(\zeta/r)^{-2d+m-2j}\int_N r^{m-1}u_j(p)\,\mathrm{dvol}_N\,\frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!}, \end{aligned}$$

hence

$$\sigma(r,\zeta)\sim_{\zeta\to\infty}\sum_{j=0}^{\infty}\zeta^{-2d+m-2j}\sigma_j(r), \tag{3.18}$$

where

$$\begin{aligned} \sigma_j(r) =& (4\pi)^{-\frac{m}{2}} r^{2d-m+2j}\int_N r^{m-1}u_j(p)\,\mathrm{dvol}_N\,\frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!} \\ =& (4\pi)^{-\frac{m}{2}} r^{2d-1+2j}\int_N u_j(p)\,\mathrm{dvol}_N\,\frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!}. \end{aligned} \tag{3.19}$$

In particular we compute $\sigma_0(r)$ and $\sigma_1(r)$. By Theorem 3.1, $u_0(p)\equiv 1$ and $u_1(p)=\frac{1}{6}\tilde{\mathrm{Scal}}(p)$, where $\tilde{\mathrm{Scal}}(p)$ is the scalar curvature on $(M,g)$, therefore

$$\sigma_0(r) = (4\pi)^{-\frac{m}{2}} r^{2d-1}\frac{\Gamma(d-\frac{m}{2})}{(d-1)!}\,\mathrm{vol}(N)$$

and

$$
\begin{aligned}
\sigma_1(r) =&(4\pi)^{-\frac{m}{2}}\frac{\Gamma(d-\frac{m}{2}+1)}{(d-1)!}r^{2d-1+2}\int_N u_1(p)\,\mathrm{dvol}_N\\
=&(4\pi)^{-\frac{m}{2}}\frac{\Gamma(d-\frac{m}{2}+1)}{(d-1)!}r^{2d+1}\int_N \frac{\tilde{\mathrm{Scal}}(p)}{6}\,\mathrm{dvol}_N\\
=&(4\pi)^{-\frac{m}{2}}\frac{\Gamma(d-\frac{m}{2}+1)}{6(d-1)!}r^{2d-1}\int_N (\mathrm{Scal}(x)-n(n-1))\,\mathrm{dvol}_N,
\end{aligned}
$$

where as before $\tilde{\mathrm{Scal}}(p)$ is the scalar curvature of $(M,g)$ at $p\in M$ and $\mathrm{Scal}(x)$ is the scalar curvature of $(N,g_N)$ at $x\in N$.

Now we show that the function $\sigma(r,\zeta)$ satisfies the three conditions of the Singular Asymptotics Lemma (Lemma 2.7). First we observe that $\sigma(r,\zeta)$ is $C^\infty$ with respect to $r$. Moreover according to [BS2, Section 3], $\sigma(r,\zeta)$ has analytic derivatives with respect to $\zeta$. The second condition of Lemma 2.7 is satisfied due to (3.18) – (3.19). The following lemma gives the proof of the third property. Denote $\partial^j\sigma(r,\zeta) := \frac{\partial^j}{\partial r^j}\sigma(r,\zeta)$.

**Lemma 3.5.** *(Integrability condition) For $j\in\mathbb{N}$ and $0<|arg\zeta|<\delta<\pi/2$ with $|\zeta|=c_0$, there is a constant $c(c_0,j)$ such that the following is satisfied uniformly for $0\le t\le 1$*

$$
\int_0^1\int_0^1 s^j\left|\partial^j\sigma(st,s\zeta)\right|dsdt\le c(c_0,j).
$$

*Proof.* Let $t\in[0,1]$. Define unitary scaling operator

$$
(U_tf)(r):=t^{\frac{1}{2}}f(tr),\qquad f\in L^2(0,\infty).
$$

Then

$$
U_tr^l=t^lr^lU_t,\qquad\qquad U_t\partial_r=t^{-1}\partial_rU_t,
$$

where $l\in\mathbb{N}$. By (3.11),

$$
U_tT=t^{-2}TU_t,
$$

note that this is true, because $T$ is the Friedrichs extension, [L, Section 2]. Define an operator in $L^2((0,\varepsilon),L^2(N))$

$$
T_t:=t^2U_tTU_t^\dagger.
$$

By Lemma 2.5, $(T_t+z^2)^{-d}$ has a continuous kernel. Denote

$$
\sigma_t(r,rz):=\mathrm{tr}_{L^2(N)}(T_t+z^2)^{-d}, \tag{3.20}
$$

in particular $\sigma_1(r, rz) = \sigma(r, rz)$.

We have $U_t(T + z^2)^{-1} = t^2(T_t + (tz)^2)^{-1}U_t$, therefore

$$U_t(T + z^2)^{-d} = t^{2d}(T_t + (tz)^2)^{-d}U_t. \tag{3.21}$$

By (3.20) and (3.21), we obtain the following scaling property

$$\sigma(rt, \zeta) = t^{2d-1}\sigma_t(r, \zeta). \tag{3.22}$$

Hence $\sigma(st, s\zeta) = t^{2d-1}\sigma_t(s, \zeta)$. By the chain rule

$$\frac{\partial}{\partial(st)}\sigma(st, s\zeta) = \frac{\partial t}{\partial(st)}\frac{\partial}{\partial t}\sigma(st, s\zeta) = s^{-1}\frac{\partial}{\partial t}\sigma(st, s\zeta),$$

therefore

$$\frac{\partial^j}{\partial(st)^j}\sigma(st, s\zeta) = s^{-j}\frac{\partial^j}{\partial t^j}\sigma(st, s\zeta).$$

Then

$$s^j\partial^j\sigma(st, s\zeta) = \partial_t^j\sigma(st, s\zeta) = \partial_t^j(t^{2d-1}\sigma_t(s, \zeta)).$$

Now choose a function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi \geq 0$ and $\varphi \equiv 1$ in the interval $[0, 1]$. Then

$$\begin{aligned}\int_0^1 s^j|\sigma^{(j)}(st, s\zeta)|ds &= \int_0^1 |\partial_t^j(t^{2d-1}\sigma_t(s, \zeta))|ds \\ &\leq ||\partial_t^j(t^{2d-1}\varphi(T + z^2)^{-d})||_{tr} \leq C_0,\end{aligned}$$

where the last inequalities follow from Lemma 2.5. □

Now we can apply the Singular Asymptotics Lemma (Lemma 2.7) to obtain

**Proposition 3.6.**

$$\begin{aligned}\int_0^\infty \varphi(r)\sigma(r, \zeta)dr \sim &\sum_{l=0}^\infty z^{-l-1}\frac{1}{l!}\fint_0^\infty \zeta^l\partial_r^l\Big(\sigma(r, \zeta)\varphi(r)\Big)|_{r=0}d\zeta \\ &+ \sum_{j=0}^\infty \fint_0^\infty \sigma_j(r)(rz)^{-2d+m-2j}\varphi(r)dr \\ &+ \sum_{l=\frac{m}{2}-d+1}^\infty z^{-2d+m-2l}\log z\frac{\partial_r^{2d-m+2l-1}\Big(\sigma_l(r)\varphi(r)\Big)|_{r=0}}{(2d-m+2l-1)!}.\end{aligned} \tag{3.23}$$

*Remark.* If $M$ is an odd-dimensional manifold, the last sum in (3.23) is zero, because $\sigma_l$ makes sense only for $l \in \mathbb{N}_0$.

Since $\varphi(r) \equiv 1$ near $r = 0$, all its derivatives vanish at zero. Therefore the first sum simplifies and we use Proposition 3.4 to obtain

$$\begin{aligned}&\sum_{l=0}^{\infty} z^{-l-1}\frac{1}{l!}\fint_0^{\infty} \zeta^l \partial_r^l \sigma(r,\zeta)|_{r=0} d\zeta \\ =&z^{-2d}\fint_0^{\infty} \zeta^{2d-1}\left(-\frac{1}{2rz}\frac{\partial}{\partial(rz)}\right)^{d-1}\sum_{\nu} I_\nu(rz)K_\nu(rz)d\zeta.\end{aligned} \tag{3.24}$$

Note that this term in the resolvent trace expansion gives a contribution to the constant term in the heat trace expansion. We simplify the terms in (3.23) in the next section.

## 3.5 The heat trace expansion

Let $f$ be a meromorphic function with the Laurent series expansion at a point $z_0$

$$f(z) = \sum_{k=-k_0}^{\infty} \operatorname{Res}_{-k} f(z_0)(z-z_0)^k,$$

where $\operatorname{Res}_{-k} f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-z_0)^{k+1}}$, for a positive oriented path $\gamma$ enclosing $z_0$ and lying in an annulus, in which $f(z)$ is holomorphic. By these notations, $\operatorname{Res}_1$ is the residue of the function and $\operatorname{Res}_0$ is the regular analytic continuation. Let $f$ be analytic at $z_0$, and assume that $g$ has a simple pole at $z_0$. Then

$$\operatorname{Res}_0(fg)(z_0) = f(z_0)\operatorname{Res}_0 g(z_0) + f'(z_0)\operatorname{Res}_1 g(z_0). \tag{3.25}$$

We analyse the first summand in Proposition 3.6 given by (3.24). Denote by $p := 2d-1$. Then by the Mellin transform [O, p.123], we obtain

$$\begin{aligned}&\fint_0^{\infty} \zeta^p\left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{d-1} I_\nu(\zeta)K_\nu(\zeta)d\zeta \\ =&\frac{1}{4\sqrt{\pi}}\frac{\Gamma(\nu-d+\frac{p+3}{2})\Gamma(d-1-\frac{p}{2})\Gamma(\frac{p+1}{2})}{\Gamma(d+1+\nu-\frac{p+3}{2})}.\end{aligned}$$

Set $p = l+2d-2$ and denote by $\operatorname{Res}_0 f(l)|_{l=1}$ the regular analytic continuation of $f(l)$ at $l = 1$, i.e. the constant term in the Laurent expansion. Following

Definition 3,

$$\begin{aligned}
&\fint_0^\infty \zeta^p \left(-\frac{1}{2\zeta}\frac{\partial}{\partial\zeta}\right)^{d-1} I_\nu(\zeta)K_\nu(\zeta)d\zeta \\
&= \mathrm{Res}_0 \left(\frac{1}{4\sqrt{\pi}}\Gamma(-\frac{l}{2})\Gamma(d+\frac{l}{2}-\frac{1}{2})\frac{\Gamma(\nu+\frac{l}{2}+\frac{1}{2})}{\Gamma(\nu-\frac{l}{2}+\frac{1}{2})}\right)|_{l=1}.
\end{aligned}$$

The ratio of two Gamma functions is given in the proposition below. Let $B_j$ be the Bernoulli numbers and $C_j^i$ be the binomial coefficients

$$B_0 = 1, B_j = -\sum_{i=0}^{j-1} C_j^i \frac{B_i}{j-i+1}, \quad j \geq 1.$$

In particular $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ and $B_{2j+1} = 0$ for $j \geq 1$.

**Proposition 3.7.** *We have that*

$$\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \sim_{\nu\to\infty} \nu^{1-2s}\left(1 + s\sum_{j\geq 1} j^{-1}B_{2j}\nu^{-2j}\right) + O(s^2).$$

*Proof.* According to [WW, Chapter XII p.251] [1],

$$\log\Gamma(z) \sim_{z\to\infty} (z-\frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \sum_{j\geq 1}\frac{1}{2j(2j-1)}\frac{B_{2j}}{z^{2j-1}}.$$

Consequently,

$$\begin{aligned}
\log\Gamma(\nu+s) \sim_{\nu\to\infty} &(\nu+s-\frac{1}{2})\log(\nu+s) - (\nu+s) + \frac{1}{2}\log(2\pi) \\
&+ \sum_{j\geq 1}\frac{1}{2j(2j-1)}\frac{B_{2j}}{(\nu+s)^{2j-1}} \\
=&(\nu+s-\frac{1}{2})(\log\nu + \log(1+s/\nu)) - (\nu+s) + \frac{1}{2}\log(2\pi) \\
&+ \sum_{j\geq 1}\frac{1}{2j(2j-1)}\frac{B_{2j}}{(\nu+s)^{2j-1}},
\end{aligned}$$

[1] Note that in this book an old notation of the Bernoulli numbers is used $B_j^{old}$, in particular $B_1^{old} = 1/6$, $B_2^{old} = 1/30$, $B_3^{old} = 1/42$; and here we use the modern notation $B_j^{new} := B_j$, they satisfy the relation $B_j^{old} = (-1)^{j-1}B_{2j}^{new}$ and $B_{2j+1}^{new} = 0$ for $j \geq 1$; that is why our formula is slightly different from the one in the book.

where for the last equality we use $\log(\nu+s) = \log\nu + \log(1+s/\nu)$. Analogously, using $\log(\nu-s) = \log\nu + \log(1-s/\nu)$, we obtain

$$\log\Gamma(\nu-s) \sim_{\nu\to\infty} (\nu - s - \frac{1}{2})(\log\nu + \log(1-s/\nu)) - (\nu-s) + \frac{1}{2}\log(2\pi) + \sum_{j\geq 1} \frac{1}{2j(2j-1)} \frac{B_{2j}}{(\nu-s)^{2j-1}}.$$

Furthermore,

$$\begin{aligned}
&\log\frac{\Gamma(\nu-s)}{\Gamma(\nu+s)} = \log\Gamma(\nu-s) - \log\Gamma(\nu+s) \sim_{\nu\to\infty} \\
&-2s\log\nu + (\nu - s - \frac{1}{2})\log(1-\frac{s}{\nu}) - (\nu + s - \frac{1}{2})\log(1+\frac{s}{\nu}) + 2s \\
&+\sum_{j\geq 1}\frac{1}{2j(2j-1)} B_{2j}\left(\frac{1}{(\nu-s)^{2j-1}} - \frac{1}{(\nu+s)^{2j-1}}\right) \\
\sim &-2s\log\nu + (\nu - s - \frac{1}{2})(-\frac{s}{\nu}) - (\nu + s - \frac{1}{2})(\frac{s}{\nu}) + 2s \\
&+\sum_{j\geq 1}\frac{1}{2j(2j-1)} B_{2j}\left(\frac{(\nu+s)^{2j-1} - (\nu-s)^{2j-1}}{(\nu^2-s^2)^{2j-1}}\right) + O(s^2) \\
= &-2s\log\nu + \frac{s}{\nu} + s\sum_{j\geq 1} j^{-1}B_{2j}\nu^{-2j} + O(s^2).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)} &= (\nu-s)\frac{\Gamma(\nu-s)}{\Gamma(\nu+s)} \\
&\sim (\nu-s)\nu^{-2s}\left(1 + \frac{s}{\nu} + s\sum_{j\geq 1} j^{-1}B_{2j}\nu^{-2j}\right) + O(s^2) \\
&\sim \nu^{1-2s}\left(1 + s\sum_{j\geq 1} j^{-1}B_{2j}\nu^{-2j}\right) + O(s^2).
\end{aligned}$$

□

Using Proposition 3.7 we compute

$$b_1 := \mathrm{Res}_0 \left( \sum_{\lambda \in \operatorname{spec} \Delta_N} \frac{\Gamma(-\frac{l}{2})\Gamma(d+\frac{l}{2}-\frac{1}{2})}{4\sqrt{\pi}(d-1)!} \frac{\Gamma(\sqrt{\lambda+(n-1)^2/4}+\frac{l}{2}+\frac{1}{2})}{\Gamma(\sqrt{\lambda+(n-1)^2/4}-\frac{l}{2}+\frac{1}{2})} \right) |_{l=1}$$
$$= \mathrm{Res}_0 \left( \frac{\Gamma(-\frac{l}{2})\Gamma(d+\frac{l}{2}-\frac{1}{2})}{4\sqrt{\pi}\Gamma(d)} \sum_{\lambda \in \operatorname{spec} \Delta_N} \left( \left(\lambda + \frac{(n-1)^2}{4}\right)^{\frac{l}{2}} \right.\right.$$
$$\left.\left. - \sum_{j \geq 1} j^{-1} B_{2j} \left(\frac{l}{2} - \frac{1}{2}\right) \left(\lambda + \frac{(n-1)^2}{4}\right)^{-j+\frac{l}{2}} \right) \right) |_{l=1}, \tag{3.26}$$

To find the regular analytic continuation at $l = 1$ of the function above we set some notations.

**Definition 4.** Let $h \in \mathbb{R}$. The *shifted by $h$ spectral zeta function* of $(N, g_N)$ is, by definition,

$$\zeta_N^h(s) := \sum_{\lambda \in \operatorname{spec} \Delta_N} \left(\lambda + h^2\right)^{-s}.$$

To compute (3.26), we need to find the residues of the shifted zeta function. Denote by $a_j^N$, $j \in \mathbb{N}_0$ the heat trace expansion coefficients on the closed manifold $(N, g_N)$

$$\operatorname{tr} e^{-t\Delta_N} \sim_{t \to 0+} (4\pi t)^{-n/2} \sum_{j=0}^{\infty} a_j^N t^j. \tag{3.27}$$

**Lemma 3.8.** *If $n$ is odd, then $\zeta_N^h(s)$ is a meromorphic function with simple poles at $s = \frac{n}{2} - l$, $l \in \mathbb{N}_0$ with residue*

$$\mathrm{Res}_1\, \zeta_N^h\left(\frac{n}{2} - l\right) = \frac{1}{(4\pi)^{n/2}\Gamma(\frac{n}{2} - l)} \sum_{i=0}^{l} (-1)^i \frac{h^{2i}}{i!} a_{l-i}^N.$$

*If $n$ is even, then the same holds, but there are no poles for $l = \frac{n}{2} + j, j \in \mathbb{N}_0$.*

*Proof.* Using the Mellin transform we obtain

$$\zeta_N^h(s) = \sum_{\lambda \in Spec\Delta_N} \left(\lambda + h^2\right)^{-s} = \frac{1}{\Gamma(s)} \sum_{\lambda \in Spec\Delta_N} \int_0^\infty t^{s-1} e^{-t(\lambda+h^2)} dt$$
$$= \frac{1}{\Gamma(s)} \sum_{\lambda \in Spec\Delta_N} \int_0^1 t^{s-1} e^{-t\lambda} e^{-th^2} dt + \frac{1}{\Gamma(s)} \sum_{\lambda \in Spec\Delta_N} \int_1^\infty t^{s-1} e^{-t(\lambda+h^2)} dt.$$

The latter integral is an entire function in $s$. To compute the first integral we use the Taylor expansion

$$e^{-th^2} \sim_{t\to 0} \sum_{i=0}^{\infty}(-1)^i \frac{h^{2i}}{i!} t^i,$$

and the heat trace expansion (3.27)). Since Gamma function is nowhere zero, we obtain

$$Res_1\zeta_N^h(s_0) = Res_1\left(\frac{1}{(4\pi)^{n/2}\Gamma(s_0)}\sum_{i=0}^{\infty}(-1)^i\frac{h^{2i}}{i!}\sum_{j=0}^{\infty}\frac{a_j^N}{(s_0+i+j-\frac{n}{2})}\right).$$

If $n$ is odd, poles are at $s_0 = \frac{n}{2} - i - j$, for $i, j \in \mathbb{N}_0$. If $n$ is even, poles are at $s_0 = \frac{n}{2} - i - j$, for $i + j < \frac{n}{2}$, because the Gamma function has simple poles at non-positive integers. Set $l := i + j$, then

$$Res_1\zeta_N^h\left(\frac{n}{2} - l\right) = \frac{1}{(4\pi)^{n/2}\Gamma(\frac{n}{2} - l)}\sum_{i=0}^{l}(-1)^i\frac{h^{2i}}{i!}a_{l-i}^N.$$

□

Note that this agrees with

**Proposition 3.9.** *[V, p.3] The spectral zeta function on $(N, g_N)$, $\zeta_N(s)$, is a meromorphic function with simple poles at $s = \frac{n}{2} - l$ for $l \in \mathbb{N}_0$ with the residue*

$$Res_1\zeta_N(s) = \frac{a^N_{\frac{n}{2}-s}}{(4\pi)^{n/2}\Gamma(s)}.$$

We now compute (3.26). Set

$$f(s) := \frac{\Gamma(\frac{s}{2})\Gamma(d - \frac{s}{2} - \frac{1}{2})}{4\sqrt{\pi}\Gamma(d)}.$$

Denote also

$$g(s) := -\sum_{j\geq 1} j^{-1}B_{2j}\zeta_N^{\frac{n-1}{2}}(j + s/2).$$

Since $f(-1) = -\frac{1}{2}$ and $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $f'(-1) = \frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(d)}{4\Gamma(d)}$, by (3.26) and (3.25), we have for $s = -l$

$$\begin{aligned} b_1 =& \operatorname{Res}_0\left(f(s)\zeta_N^{\frac{n-1}{2}}(s/2) + f(s)\left(-\frac{s}{2} - \frac{1}{2}\right)g(s)\right) \\ =& -\frac{1}{2}\operatorname{Res}_0\zeta_N^{\frac{n-1}{2}}(-1/2) \\ &+ \left(\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(d)}{4\Gamma(d)}\right)\operatorname{Res}_1\zeta_N^{\frac{n-1}{2}}(-1/2) + \frac{1}{4}\operatorname{Res}_1 g(-1). \end{aligned}$$

By Lemma 3.8,

$$
\begin{aligned}
b_1 = & -\frac{1}{2}\operatorname{Res}_0 \zeta_N^{\frac{n-1}{2}}(-1/2) + \left(\frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} + \frac{\Gamma'(d)}{4\Gamma(d)}\right)\operatorname{Res}_1 \zeta_N^{\frac{n-1}{2}}(-1/2) \\
& -\frac{1}{4}\sum_{j=1}^{\frac{n+1}{2}} j^{-1}B_{2j}\operatorname{Res}_1 \zeta_N^{\frac{n-1}{2}}(j-1/2).
\end{aligned}
\tag{3.28}
$$

If $n$ is even, the above sum is to be interpreted as sum over all integers $1 \le j \le \frac{n+1}{2}$.

Now consider the second sum in the expansion (3.23). By (3.19),

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \fint_0^{\infty} \sigma_j(r)(rz)^{-2d+m-2j}\varphi(r)dr \\
=&(4\pi)^{-\frac{m}{2}}\sum_{j=0}^{\infty}\frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!}z^{-2d+m-2j}\fint_0^{\infty} r^{m-1}\varphi(r)\int_N u_j(p)\,\mathrm{dvol}_N\,dr.
\end{aligned}
\tag{3.29}
$$

By [BGM, Chapter III, Lemma E.IV.5], $u_j(r,x) = r^{-2j}\hat{u}_j(r,x)$, where $\hat{u}_j(r,x)$ is smooth with respect to $r$. Hence for $m-1-2j \ge 0$, the integrals in the above sum need no regularisation, i.e. for $j \le m/2-1$ we have

$$
\fint_0^{\infty} r^{m-1}\varphi(r)\int_N u_j(p)\,\mathrm{dvol}_N\,dr = \int_0^{\infty} r^{m-1}\varphi(r)\int_N u_j(p)\,\mathrm{dvol}_N\,dr.
$$

The sum (3.29) gives the following terms in the heat trace expansion

$$
\begin{aligned}
&(4\pi)^{-\frac{m}{2}}\sum_{j=0}^{\infty} t^{-\frac{m}{2}+j}\fint_0^{\infty} r^{m-1}\varphi(r)\int_N u_j(p)\,\mathrm{dvol}_N\,dr \\
=\ &(4\pi)^{-\frac{m}{2}}\sum_{j=0}^{\infty} t^{-\frac{m}{2}+j}\fint_M u_j(p)\varphi(r)\,\mathrm{dvol}_M,
\end{aligned}
\tag{3.30}
$$

where summands with $j \le m/2-1$ need no regularisation.

We sum this with the expansion away from the singularity to obtain

$$
\begin{aligned}
&(4\pi)^{-\frac{m}{2}}\sum_{j=0}^{\infty} t^{-\frac{m}{2}+j}\left(\fint_M u_j(p)\varphi(r)\,\mathrm{dvol}_M + \int_M u_j(p)(1-\varphi(r))\,\mathrm{dvol}_M\right) \\
&= \sum_{j=0}^{\infty} t^{-\frac{m}{2}+j}\fint_M u_j(p)\,\mathrm{dvol}_M\,.
\end{aligned}
$$

Now we simplify the logarithmic terms in Proposition 3.6

$$
\begin{aligned}
L :=& \sum_{j=\frac{m}{2}-d+1}^{\infty} z^{-2d+m-2j} \log z \frac{\partial_r^{2d-m+2j-1}\Big(\sigma_j(r)\varphi(r)\Big)|_{r=0}}{(2d-m+2j-1)!} \\
=& \sum_{j=\frac{m}{2}-d+1}^{\infty} z^{-2d+m-2j} \log z \frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!(2d-m+2j-1)!} \times \\
& \times (4\pi)^{-\frac{m}{2}} \partial_r^{2d-m+2j-1}\left(r^{2d-1+2j}\int_N u_j(p)\,\mathrm{dvol}_N\right)|_{r=0}.
\end{aligned}
$$

To continue the computation we note that for $j \geq m/2$ we have

$$
\begin{aligned}
&\partial_r^{2d-m+2j-1}\left(r^{2d-1+2j}\int_N u_j(p)\,\mathrm{dvol}_N\right)|_{r=0} \\
&\qquad = \partial_r^{2j-m}\left(r^{2j}\int_N u_j(p)\,\mathrm{dvol}_N\right)|_{r=0},
\end{aligned}
$$

therefore

$$
\begin{aligned}
L =& \sum_{j=\frac{m}{2}}^{\infty} z^{-2d+m-2j} \log z \frac{\Gamma(-\frac{m}{2}+d+j)}{(d-1)!(2j-m)!} \times \\
& \times (4\pi)^{-\frac{m}{2}} \partial_r^{2j-m}\left(r^{2j}\int_N u_j(p)\,\mathrm{dvol}_N\right)|_{r=0} \\
=& \sum_{l=0}^{\infty} z^{-2d-2l} \log z \frac{\Gamma(d+l)}{(d-1)!(2l)!} (4\pi)^{-\frac{m}{2}} \partial_r^{2l}\left(r^{m+2l}\int_N u_{\frac{m}{2}+l}(p)\,\mathrm{dvol}_N\right)|_{r=0}.
\end{aligned}
$$

Hence the logarithmic part in the heat trace expansion coming from this term is

$$
-(4\pi)^{-\frac{m}{2}} \sum_{i=0}^{\infty} t^i \log t \times \frac{1}{2(2i)!} \partial_r^{2i}\left(r^{m+2i}\int_N u_{\frac{m}{2}+i}(p)\,\mathrm{dvol}_N\right)|_{r=0}.
$$

For $i > 0$, the function $r^{m+2i}\int_N u_{\frac{m}{2}+i}(p)\,\mathrm{dvol}_N$ is smooth and has no terms of order $r^{2i}$. Hence the only nonzero logarithmic term may appear for $i = 0$

$$
c := -(4\pi)^{-\frac{m}{2}} \frac{1}{2}\left(r^m \int_N u_{\frac{m}{2}}(p)\,\mathrm{dvol}_N\right)|_{r=0} = \frac{1}{4}\operatorname{Res}_1 \zeta_N^{\frac{n-1}{2}}(-1/2). \tag{3.31}
$$

From the logarithmic term in the resolvent trace expansion we also get the following contribution to the constant term in the heat trace expansion

$$b_2 = (4\pi)^{-\frac{m}{2}} \frac{1}{2} \frac{\Gamma'(d)}{\Gamma(d)} \left( r^m \int_N u_{\frac{m}{2}}(p) dvol_N \right)|_{r=0} = -\frac{1}{4} \frac{\Gamma'(d)}{\Gamma(d)} Res_1 \zeta_N^{\frac{n-1}{2}}(-1/2). \tag{3.32}$$

Let $b := b_1 + b_2$. By (3.28) and (3.32),

$$\begin{aligned} b = &-\frac{1}{2} \operatorname{Res}_0 \zeta_N^{\frac{n-1}{2}}(-1/2) + \frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}} \operatorname{Res}_1 \zeta_N^{\frac{n-1}{2}}(-1/2) \\ &- \frac{1}{4} \sum_{j=1}^{\frac{n+1}{2}} j^{-1} B_{2j} \operatorname{Res}_1 \zeta_N^{\frac{n-1}{2}}(j-1/2). \end{aligned} \tag{3.33}$$

## 3.6 Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* First, we show that if $m \geq 4$, then the Laplace-Beltrami operator on $(M, g)$ is essentially self-adjoint. By (3.11),

$$\Delta = -\partial_r^2 + r^{-2} \left( \frac{n}{2} \left( \frac{n}{2} - 1 \right) + \Delta_N \right),$$

where $n = m - 1$ and $\Delta_N$ is the Laplace-Beltrami operator on $(N, g_N)$. By [BS3, pp.703-704], $\Delta$ is essentially self-adjoint if

$$\left( \frac{n}{2} \left( \frac{n}{2} - 1 \right) + \Delta_N \right) \geq \frac{3}{4},$$

equivalently,

$$n \geq 3,$$

otherwise $\Delta$ might have many self-adjoint extensions.

(a) By (3.30), (3.31) and (3.33), we obtain the final formula

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to 0+} (4\pi t)^{-\frac{m}{2}} \sum_{j=0}^{\infty} \tilde{a}_j t^j + b + c \log t. \tag{3.34}$$

Above for $j \leq m/2 - 1$, we have $\tilde{a}_j = \int_M u_j(p) \, \mathrm{dvol}_M$. For $j > m/2 - 1$, we have the reguralized integrals $\tilde{a}_j = ⨍_M u_j(p) \, \mathrm{dvol}_M$.

(b) The constant term $b$ is given by (3.33).

By Lemma 5.7, the constant term $b_{S^n}$ in the heat trace expansion on $(M_1, g_1)$ with the cross-section $(N, g_N) = (S^n, g_{\text{round}})$ is equal to zero. By Theorem 5.8, the constant term $b_{\mathbb{R}P^n}$ in the heat trace expansion on $(M_2, g_2)$ with the cross-section $(N, g_N) = (\mathbb{R}P^n, g_{\text{round}})$ is non-zero. Observe, that $S^n$ is the covering of $\mathbb{R}P^n$. By the first claim of this theorem, $\tilde{a}_j$ has an expression by the integral of the local data that is the curvature and its derivatives. Assume that the constant term $b$ can be written as an integral over local data. Then $b$ satisfies the multiplicative law for coverings, hence $b_{S^n} = 2b_{\mathbb{R}P^n}$. It is a contradiction. We conclude that in general there is no expression of $b$ as an integral of local data.

(c) By (3.31), we get

$$c = \frac{1}{4} \operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(-1/2).$$

By Lemma 3.8,

$$\operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(s) = \operatorname{Res}_1 \left( \frac{1}{(4\pi)^{\frac{m-1}{2}} \Gamma(s)} \sum_{k=0}^{\infty} (-1)^k \frac{(m-2)^{2k}}{2^{2k} k!} \sum_{j=0}^{\infty} \frac{a_j^N}{(s + k + j - \frac{m-1}{2})} \right).$$

Setting $s = -\frac{1}{2}$, we get the relation between the summation indices

$$k + j = \frac{m}{2},$$

hence

$$\operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(-1/2) = \frac{1}{(4\pi)^{\frac{m-1}{2}} \Gamma(-\frac{1}{2})} \sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{(m-2)^{2k}}{2^{2k} k!} a^N_{\frac{m}{2}-k}, \tag{3.35}$$

and we simplify

$$\begin{aligned} \operatorname{Res}_1 \zeta_N^{\frac{m-2}{2}}(-1/2) =& -\frac{1}{(4\pi)^{\frac{m}{2}}} \sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{(m-2)^{2k}}{2^{2k} k!} a^N_{\frac{m}{2}-k} \\ =& \frac{1}{(4\pi)^{\frac{m}{2}}} \sum_{k=0}^{\frac{m}{2}} (-1)^{k+1} \frac{(m-2)^{2k}}{2^{2k} k!} a^N_{\frac{m}{2}-k}. \end{aligned}$$

If $m$ is odd we have $a^N_{\frac{m}{2}-k} = 0$ for $0 \le k < \frac{m}{2}$.

(d) Assume that $c = 0$, then by (3.31),

$$\left( r^m \int_N u_{\frac{m}{2}}(r, x) \operatorname{dvol}_N \right)|_{r=0} = 0.$$

Hence

$$
\begin{aligned}
\tilde{a}_{\frac{m}{2}} &= \fint_M u_{\frac{m}{2}}(p)\,\mathrm{dvol}_M = \fint_U u_{\frac{m}{2}}(p)\,\mathrm{dvol}_M + \int_{M\setminus U} u_{\frac{m}{2}}(p)\,\mathrm{dvol}_M \\
&= \int_{M\setminus U} u_j(p)\,\mathrm{dvol}_M = \int_M u_j(p)\,\mathrm{dvol}_M\,.
\end{aligned}
$$

This finishes the proof. □

# 4 Geometrical information from the heat trace expansion

In this section we analyse the coefficients in the expansion in Theorem 1.1 from a geometrical point of view. First, we consider the two-dimensional case and show in which setting we can distinguish a surface with conic singularities from a compact smooth surface using the heat trace expansion.

Next, we consider the four-dimensional case and develop a criterion of the cross-section $(N, g_N)$ near a singularity to be isometric to a spherical space form. Then we study out some illustrative four-dimensional examples.

We complete the section with a geometrical criterion for the logarithmic term in the heat trace expansion in Theorem 1.1 to be equal to zero that holds for any even-dimensional $(M, g)$. Recall that for any odd-dimensional $(M, g)$ the logarithmic term is identically zero.

## 4.1 Compact surfaces with conic singularities

The constant term in the heat trace expansion on surfaces with conic singularities was computed in [BS2, pp.423–424]. Here we prove that the logarithmic term in the expansion is zero and $\tilde{a}_j$, in Theorem 1.1, does not need regularization for any $j \geq 0$.

Consider a surface $(M, g)$ with $l$ conic singularities. By this we mean $(M, g)$, which possesses an open set $\cup_{i=1}^{l} U_i$ such that $M \setminus \cup_{i=1}^{l} U_i$ is a compact surface with boundary $N_1 \cup \cdots \cup N_l$. Furthermore, $U_i$ is isometric to $(0, \varepsilon) \times N_i$ with $\varepsilon > 0$, where $(N_i, g_{N_i})$ is a closed one-dimensional manifold for $1 \leq i \leq l$. The metric in the neighbourhood $U_i$ is

$$g_{U_i} = dr^2 + r^2 g_{N_i},$$

where $g_{N_i} := \sin^2 \alpha_i d\theta$. Here $0 < \theta \leq 2\pi$ is the local coordinate on $S^1$ and $0 < \alpha_i \leq \pi/2$ is the angle between the generating line and the axis of the cone $U_i$.

**Lemma 4.1.** *The heat trace expansion on a surface $(M, g)$ with $l$ conic singularities has the following form*

$$\operatorname{tr} e^{-t\Delta} \sim_{t \to 0+} \frac{1}{4\pi t} \sum_{j=0}^{\infty} a_j t^j + \frac{1}{12} \sum_{i=1}^{l} \left( \frac{1}{\sin \alpha_i} - \sin \alpha_i \right), \tag{4.1}$$

*where $a_j$ does not have any contribution from the singularities for $j \geq 0$.*

*Proof.* By Theorem 1.1, the heat trace expansion on $M$ is

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to 0+} \frac{1}{4\pi t}\sum_{j=0}^{\infty} \tilde{a}_j t^j + b + c\log t, \tag{4.2}$$

where $\tilde{a}_j$ are integrals over $(M,g)$, which might need regularization.

First, we show that $c = 0$. Observe that every singularity contributes to the logarithmic term. The scalar curvature of a circle is zero, hence by Theorem 1.1 (c),

$$c = \sum_{i=1}^{l} c_i = -\frac{1}{4(4\pi)}\sum_{i=1}^{l} a_1^{N_i} = -\frac{1}{16\cdot 6\pi}\sum_{i=1}^{l}\int_{N_i} \operatorname{Scal} \operatorname{dvol}_{N_i} = 0.$$

Next, we compute the constant term in the heat trace expansion (4.2). Since $\operatorname{Spec}\Delta_{N_i} = \{\sin\alpha_i^{-2}k^2 \mid k\in\mathbb{Z}\}$, the spectral zeta function on $N_i$ is

$$\zeta^0_{N_i}(s) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{k}{\sin\alpha_i}\right)^{-2s} = 2\sin\alpha_i^{2s}\zeta(2s),$$

where $\zeta(s)$ is the Riemann zeta function.

By the regular analytic continuation of the Riemann zeta function, we obtain

$$\operatorname{Res}_0 \zeta^0_{N_i}(-1/2) = -2\sin\alpha_i^{-1}\frac{1}{12} = -\frac{1}{6\sin\alpha_i}.$$

The residues are

$$\operatorname{Res}_1 \zeta^0_{N_i}(j-1/2) = \operatorname{Res}\zeta(2j-1) = 0, \text{ for } j\geq 2$$

and

$$\operatorname{Res}_1 \zeta^0_{N_i}(1/2) = 2\sin\alpha_i \operatorname{Res}\zeta(1) = 2\sin\alpha_i.$$

Thus, the constant term in the heat trace expansion is

$$b = \sum_{i=1}^{l} b_i = \sum_{i=1}^{l}\left(\frac{1}{12\sin\alpha_i} - \frac{1}{4}B_2 2\sin\alpha_i\right).$$

Hence

$$b = \frac{1}{12}\sum_{i=1}^{l}\left(\frac{1}{\sin\alpha_i} - \sin\alpha_i\right).$$

Next, we compute coefficients $\tilde{a}_j$ in (4.2). By Theorem 1.1 (a),

$$\tilde{a}_j = ⨍_M u_j(p)\operatorname{dvol}_M = \int_M \psi u_j(p)\operatorname{dvol}_M + \sum_{1\leq i\leq l} ⨍_M \varphi_i(r)u_j(p_i)\operatorname{dvol}_M,$$

where $\psi, \varphi_1(r), \ldots \varphi_l(r)$ is a partition of unity on $(M, g)$ such that $\psi$ has support away from the singular points and $\varphi_i$ has support in $(U_i, g_{U_i})$.

The curvature tensor on the cone over a circle is equal to zero, hence by Theorem 3.1 we have $u_j(p) \equiv 0$ for $p \in U_i$ for $1 \leq i \leq l$ and $j > 0$, also $u_0(p) \equiv 1$ for $p \in M$. Therefore, the singularities do not contribute to the positive power terms in the expansion and

$$\tilde{a}_j = \begin{cases} \mathrm{vol}(M) \text{ for } i = 0, \\ \int_M u_j(p)\, \mathrm{dvol}_M, \text{ for } j > 0. \end{cases}$$

Above neither $\tilde{a}_j$ needs regularization, i.e. $\tilde{a}_j$ is equal to $a_j$ in (1.2). We conclude that the geometric information about the singularities is encoded only in the constant term. Finally,

$$\operatorname{tr} e^{-t\Delta} \sim_{t \to 0+} \frac{1}{4\pi t} \sum_{j=0}^{\infty} a_j t^j + \frac{1}{12} \sum_{i=1}^{l} \left( \frac{1}{\sin \alpha_i} - \sin \alpha_i \right). \tag{4.3}$$

□

**Definition 5.** Two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ are called *isospectral* if the eigenvalues of their Laplace-Beltrami operators coincide.

Denote by $(\bar{M}, g)$ the complete surface with conic singularities such that the conic points belong to $(\bar{M}, g)$, i.e. the neighbourhood of a conic point, $U_i'$, is isometric to $[0, \varepsilon) \times N_i$ for $1 \leq i \leq l$.

**Theorem 4.2.** *Let $(\bar{M}, g)$ be a complete simply-connected surface with conic singularities. If $(\bar{M}, g)$ has at least one singularity, then it is not isospectral to any smooth closed surface.*

*Proof.* By Lemma 4.1,

$$\begin{aligned} \operatorname{tr} e^{-t\Delta} \sim_{t \to 0+} & \frac{1}{24\pi} \int_M \tilde{\mathrm{Scal}}(p)\, \mathrm{dvol}_M \\ & + \frac{1}{12} \sum_{i=1}^{l} \left( \frac{1}{\sin \alpha_i} - \sin \alpha_i \right) + \frac{1}{4\pi t} \sum_{j \geq 0, j \neq 1} a_j t^j, \end{aligned}$$

where $\tilde{\mathrm{Scal}}(p)$ is the scalar curvature at $p \in M$.

By [BL, pp.52-53],

$$\frac{1}{2} \int_M \tilde{\mathrm{Scal}}(p)\, \mathrm{dvol}_M = 2\pi(2 - 2\, \mathrm{genus}(\bar{M})).$$

Since $(\bar{M}, g)$ is simply-connected, $\mathrm{genus}(\bar{M}) = 0$ and

$$\operatorname{tr} e^{-t\Delta} \sim_{t\to 0+} \frac{1}{3} + \frac{1}{12}\sum_{i=1}^{l}\left(\frac{1}{\sin\alpha_i} - \sin\alpha_i\right) + \frac{1}{4\pi t}\sum_{j\geq 0, j\neq 1} a_j t^j.$$

Observe that

$$\frac{1}{\sin\alpha_i} - \sin\alpha_i \geq 0,$$

and the equality holds if and only if $\alpha_i = \frac{\pi}{2}$, i.e. the corresponding neighbourhood $U_i'$ of the singularity is a disk. Therefore, if $(\bar{M}, g)$ has at least one singularity, we obtain the equality for the constant coefficient in the heat trace expansion

$$\frac{1}{3} + \frac{1}{12}\sum_{i=1}^{l}\left(\frac{1}{\sin\alpha_i} - \sin\alpha_i\right) > \frac{1}{3}.$$

Let $(M_1, g_1)$ be a smooth closed surface and $\Delta_{M_1}$ be the Laplace-Beltrami operator on it. Then by (1.2), the heat trace expansion on $(M_1, g_1)$ has the following form

$$\operatorname{tr} e^{-t\Delta_{M_1}} \sim_{t\to 0+} \frac{1}{3}(1 - \mathrm{genus}(M_1)) + \frac{1}{4\pi t}\sum_{j\geq 0, j\neq 1} a_j' t^j.$$

The constant coefficient in the heat trace expansion satisfies the following equality

$$\frac{1}{3}(1 - \mathrm{genus}(M_1)) \leq \frac{1}{3}.$$

We conclude that since the constant term in the heat trace expansion on $(\bar{M}, g)$ with at least one conic singularity is grater than $\frac{1}{3}$, $(\bar{M}, g)$ and $(M_1, g_1)$ cannot be isospectral. □

## 4.2 Four-dimensional manifolds with conic singularities

In this section we consider the spectral geometry of four-dimensional manifolds with conic singularities. As before, we use symbols with tildes to denote tensors on $(M, g)$, and without tildes to denote tensors on the cross-section $(N, g_N)$.

**Theorem 4.3.** *Let $(M, g)$ be a four-dimensional manifold with conic singularities.*

*(1) The logarithmic term in the heat trace expansion on $(M,g)$ is equal to zero if and only if the cross-section of every singularity is isometric to a spherical space form.*

*(2) Assume that the logarithmic term in the heat trace expansion on $(M,g)$ is equal to zero. Then no regularization is needed for $\tilde{a}_j$, for any $j \geq 0$. In this case only the constant term in the heat trace expansion has a contribution from the singularities.*

*Proof.* We consider one conic singularity and show that its contribution to the logarithmic term is non-positive. Moreover the contribution is equal to zero if and only if the cross-section is isometric to a spherical space form. Consequently, if $(M,g)$ has more conic singularities, the logarithmic term is equal to zero if and only if each cross-section is isometric to a spherical space form. Similarly, we show that the second claim of the theorem holds for a manifold with one conic singularity, i.e. we do not need to regularise integrals in $\tilde{a}_j$, hence it holds for more singularities.

As before (3.27), we denote the heat trace coefficients on $(N, g_N)$ by $a_j^N$, $j \geq 0$. By Theorem 1.1 (c),

$$c = \frac{1}{4(4\pi)^2} \sum_{k=0}^{2} (-1)^{k+1} \frac{1}{k!} a_{2-k}^N = \frac{1}{4(4\pi)^2}(-a_2^N + a_1^N - \frac{1}{2} a_0^N).$$

We use Theorem 3.1 and (3.1), to obtain

$$\begin{aligned} c = &- \frac{1}{720}(4\pi)^{-2} \int_N (12\Delta_N \operatorname{Scal}(x) + 5(\operatorname{Scal}(x) - 6)^2 - 2|\operatorname{Ric}(x)|^2 \\ &+ 2|\mathrm{R}(x)|^2) \operatorname{dvol}_N . \end{aligned} \tag{4.4}$$

Since $(N, g_N)$ is closed, for any smooth function $\tau \in C^\infty(N)$ we have

$$\begin{aligned} \int_N \Delta_N \tau \operatorname{dvol}_N &= \int_N (d^\dagger d\tau) \operatorname{dvol}_N = \int_N (d^\dagger d\tau, 1)_x \operatorname{dvol}_N \\ &= \int_N (d\tau, d1)_x \operatorname{dvol}_N = \int_N (d\tau, 0)_x \operatorname{dvol}_N = 0, \end{aligned}$$

where $(\cdot,\cdot)_x$ denotes the inner product on the tangent space $T_xM$. Hence (4.4) becomes

$$c = - \frac{1}{720}(4\pi)^{-2} \int_N (5(\operatorname{Scal}(x) - 6)^2 - 2|\operatorname{Ric}(x)|^2 + 2|\mathrm{R}(x)|^2) \operatorname{dvol}_N . \tag{4.5}$$

For $\dim N = 3$, the Ricci curvature tensor determines the Riemann curvature tensor, and we have the identity at every point $x \in N$

$$|\mathrm{R}(x)|^2 = 4|\mathrm{Ric}(x)|^2 - \mathrm{Scal}^2(x). \tag{4.6}$$

We compute (4.5) using (4.6),

$$\begin{aligned}
c =& -\frac{1}{720}(4\pi)^{-2} \int_N \Big(5(\mathrm{Scal}(x) - 6)^2 - 2\,\mathrm{Scal}^2(x) + 6|\mathrm{Ric}(x)|^2\Big)\,\mathrm{dvol}_N \\
=& -\frac{1}{720}(4\pi)^{-2} \int_N \Big(5(\mathrm{Scal}(x) - 6)^2 - 2\,\mathrm{Scal}^2(x) + 24\,\mathrm{Scal}(x) - 2\cdot 36 \\
&+ 6(|\mathrm{Ric}(x)|^2 - 4\,\mathrm{Scal}(x) + 12\Big)\,\mathrm{dvol}_N \\
=& -\frac{1}{720}(4\pi)^{-2} \int_N \Big(3(\mathrm{Scal}(x) - 6)^2 + 6(\mathrm{Ric}_{ij}(x) - 2g_{ij})^2\Big)\,\mathrm{dvol}_N \leq 0.
\end{aligned}$$

This expression is equal to zero if and only if $\mathrm{Scal}(x) \equiv 6$ and $\mathrm{Ric}_{ij}(x) = 2g_{ij}(x)$; equivalently if and only if the sectional curvature of $(N, g_N)$ is equal to one. Therefore $(N, g_N)$ is isometric to a spherical space form. We have proved the first part of the theorem.

(2) If the logarithmic term is equal to zero, then by the first claim of the theorem, the neighbourhood of the singularity $(U, g_U)$ has the following metric

$$g_U = dr^2 + r^2 g_{S^3},$$

where $g_{S^3}$ is the round metric on the unit sphere. Thus $(U, g_U)$ is isometric to the punctured ball. The ball is a flat space, i.e. the curvature tensor vanishes for every $p \in U$, hence $u_j(p) = 0$ for $j > 0$ and $u_0(p) \equiv 1$. Therefore,

$$\tilde{a}_j = \fint_M u_j(p)\,\mathrm{dvol}_M = \begin{cases} \mathrm{vol}(M), & \text{for } j = 0, \\ \int_M u_j(p)\,\mathrm{dvol}_M, & \text{for } j > 0. \end{cases}$$

Therefore no regularization is needed for $\tilde{a}_j$, for any $j \geq 0$, and the only term that has a contribution of the singularity is the constant term $b$.

$\square$

Assume that $(N, g_N)$ is isometric to a spherical space form and compute the constant term $b$ in the heat trace expansion on $(M, g)$. Since $c = 0$, we have $\mathrm{Res}_1 \zeta_N^1(-1/2) = 0$. By Theorem 1.1 (b),

$$b = -\frac{1}{2}\zeta_N^1(-1/2) - \frac{1}{4}B_2\,\mathrm{Res}_1\,\zeta_N^1(1/2) - \frac{1}{8}B_4\,\mathrm{Res}_N^1(3/2). \tag{4.7}$$

By Lemma 3.8,

$$\operatorname{Res}_1 \zeta_N^1(3/2-l) = \frac{1}{(4\pi)^{\frac{3}{2}}\Gamma(\frac{3}{2}-l)} \sum_{i=0}^{l} \frac{(-1)^i}{i!} a_{l-i}^N.$$

Using $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$ and $\mathrm{Scal}(x) \equiv 6$, we compute

$$\begin{aligned}\operatorname{Res}_1 \zeta_N^1(1/2) =& \frac{1}{(4\pi)^{\frac{3}{2}}\Gamma(\frac{1}{2})}(a_1^N - a_0^N) \\ =& \frac{2}{(4\pi)^2}\left(\frac{1}{6}\int_N \mathrm{Scal}(x)\,\mathrm{dvol}_N - \mathrm{vol}(N)\right) = 0,\end{aligned}$$

and

$$\operatorname{Res}_1 \zeta_N^1(3/2) = \frac{1}{(4\pi)^{\frac{3}{2}}\Gamma(\frac{3}{2})}(a_0^N) = \frac{1}{4\pi^2}\mathrm{vol}(N).$$

Then (4.7) becomes

$$b = -\frac{1}{2}\zeta_N^1(-1/2) + \frac{1}{960\pi^2}\mathrm{vol}(N).$$

In the four-dimensional case, the logarithmic term in the expansion vanishes precisely when the cross-section is a spherical space form, and we expect that the vanishing of a the constant term will imply that the cross-section is the sphere $(S^3, g_{\mathrm{round}})$, but this is not yet clear beyond the case of lens spaces. The constant term $b$ was computed for lens spaces $(S^3/\mathbb{Z}^k, g_{\mathrm{round}})$, see [DB, pp. 2281–2282], also [BKD, (8.4)]

$$b = \frac{(k^2+11)(k^2-1)}{720k}. \tag{4.8}$$

We observe that $b = 0$ if and only if $k = 1$. However there are other types of spherical space forms for which the expression of the constant term is not clear yet.

### 4.2.1 Example $N = S_A^3$

Consider a four-dimensional manifold $(M, g)$ with one conic singularity and the following metric near the singularity,

$$g_{\mathrm{conic}} = dr^2 + r^2 g_N.$$

Above $(N, g_N)$ is the three-dimensional sphere $N = S^3_A$ with radius $A > 0$. We compute the logarithmic term in the heat trace expansion on $M$ and show that it is equal to zero if and only if $A = 1$.

The metric on the sphere is given by

$$g_N = A^2(d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \sin^2\varphi_1 \sin^2\varphi_2 d\varphi_3^2),$$

where $0 \leq \varphi_i < 2\pi$ for $i = 1, 2, 3$, and the volume $\mathrm{vol}(S^3_A) = 2\pi^2 A^3$.

We compute the norm of the Ricci curvature tensor on $S^3_A$

$$|\mathrm{Ric}(x)|^2 = \left|2\frac{1}{A^2}g_{ij}(x)\right|^2 = 4\frac{1}{A^4}g_{ij}(x)g_{ij}(x)g^{ii}(x)g^{jj}(x) = \frac{12}{A^4},$$

and the scalar curvature

$$\mathrm{Scal}(x) = \frac{6}{A^2}.$$

By (4.5), the logarithmic term is

$$\begin{aligned} c =& -\frac{1}{720}(4\pi)^{-2}\int_{S^3_A}\left[5(\mathrm{Scal}(x) - 6)^2 + 6|\mathrm{Ric}(x)|^2 - 2\,\mathrm{Scal}(x)^2\right]\mathrm{dvol}_{S^3_A} \\ =& -\frac{1}{720}(4\pi)^{-2}\frac{5\cdot 36(A^2-1)^2}{A^4}v\,\mathrm{vol}(S^3_A) = -\frac{(A^2-1)^2}{32A}. \end{aligned}$$

We see that the logarithmic term in the heat trace expansion is zero if and only if $A = 1$, i.e. the sphere has a unit radius. To compute the constant term for $S^3_1$, we need the eigenvalues on the sphere.

**Proposition 4.4** ( [BGM, Prop.C.I.1, Cor.C.I.3, p.160] )**.** *The sphere $(S^n, g_{round})$ has eigenvalues*

$$\lambda_k = k(n + k - 1), \quad k \geq 0,$$

*with multiplicities*

$$\mu_k = \frac{(n+k-2)(n+k-3)\dots(n+1)n}{k!}(n + 2k - 1).$$

In our case $n = 3$, and the eigenvalues are

$$\lambda_k = k(k + 2),$$

with corresponding multiplicities

$$\mu_k = (k + 1)^2.$$

Describe the shifted zeta function

$$
\begin{aligned}
\zeta_N^1(s) &= \sum_{k\in\mathbb{N}_0} \mu_k \left(\lambda_k+1\right)^{-s} \\
&= \sum_{k\in\mathbb{N}_0} (k+1)^2 \left(k^2+2k+1\right)^{-s} = 1 + \sum_{k\in\mathbb{N}} (k+1)^{-2s+2} \\
&=1 + \sum_{k\in\mathbb{N}} k^{-2s+2}\Bigg(1 + (-2s+2)k^{-1} + \frac{(-2s+2)(-2s+1)}{2}k^{-2} \\
&\quad + \frac{(-2s+2)(-2s+1)(-2s)}{6}k^{-3} \\
&\quad + \frac{(-2s+2)(-2s+1)(-2s)(-2s-1)}{24}k^{-4} + \sum_{v=5}^{\infty}\binom{-2s+2}{v}k^{-v}\Bigg).
\end{aligned}
$$

The last equality is due to the generalized binomial theorem and $\binom{r}{v} = \frac{r(r-1)\dots(r-v+1)}{v!}$. Hence

$$
\begin{aligned}
\zeta_N^1(s) =&1 + \zeta(2s-2) - (2s-2)\zeta(2s-1) + \frac{(2s-2)(2s-1)}{2}\zeta(2s) \\
&- \frac{(2s-2)(2s-1)2s}{6}\zeta(2s+1) \\
&+ \frac{(2s-2)(2s-1)2s(2s+1)}{24}\zeta(2s+2) + \rho(s).
\end{aligned}
$$

Above $\zeta(s)$ is the Riemann zeta function, and $\rho(s)$ is analytic for $s > -2$. We compute the residues

$$
\operatorname{Res}_1 \zeta_N^1(-1/2) = 0 = \operatorname{Res}_1 \zeta_N^1(1/2)
$$

and

$$
\operatorname{Res}_1 \zeta_N^1(3/2) = 1.
$$

Using $\zeta(-3) = \frac{1}{120}$, $\zeta(-1) = -\frac{1}{12}$ and $\zeta(0) = -\frac{1}{2}$, we obtain

$$
\operatorname{Res}_0 \zeta_N^1(-1/2) = 1 + \frac{1}{120} - 3\frac{1}{12} - \frac{1}{2} - \frac{1}{4} = \frac{1}{120}.
$$

By (3.33),

$$
b = -\frac{1}{2}\frac{1}{120} + \frac{1}{8}\frac{1}{30} = 0.
$$

This example illustrates that for $N = S_A^3$ the logarithmic term in the heat trace expansion vanishes if and only if $A = 1$. Moreover, for $N = S_1^3$ the constant term $b$ vanishes as well and the heat trace expansion on $M$ looks exactly like the heat trace expansion on a smooth compact manifold.

### 4.2.2 Example $N = \mathbb{R}P^3$

Consider a four-dimensional manifold $(M, g)$ with one conic singularity and the following metric near the singularity

$$g_{\text{conic}} = dr^2 + r^2 g_N,$$

where $N = \mathbb{R}P^3$ with the round metric. We show that the logarithmic term in the heat trace expansion on $(M, g)$ is equal to zero and compute the constant term $b$.

Note that $N = \mathbb{R}P^3$ is a spherical space form, hence the metric $g_{\text{conic}}$ is locally flat and the curvature tensors are identically zero.

By Theorem 1.4, we obtain

$$c = 0.$$

To compute the constant term we need information about the eigenvalues of the Laplace-Beltrami operator on $\mathbb{R}P^3$.

**Proposition 4.5** ( [BGM, Prop.C.II.1, p.166] )**.** *Let $(\mathbb{R}P^n, g_{round})$ be the projective space. The Laplace-Beltrami operator has the following eigenvalues*

$$\lambda_k = 2k(n + 2k - 1), \quad k \geq 0,$$

*with multiplicities*

$$\frac{(2k+n-2)!}{(2k)!(n-1)!}(n + 4k - 1).$$

For $n = 3$, we have $\lambda_k = 4k(k + 1)$ with multiplicity $(2k + 1)^2$, therefore the shifted spectral zeta function

$$\begin{aligned}
\zeta_N^1(s) = &\sum_{k\in\mathbb{N}_0}(2k+1)^2(\lambda_k+1)^{-s} = \sum_{k\in\mathbb{N}_0}(2k+1)^2(2k+1)^{-2s} \\
=&1 + \sum_{k\in\mathbb{N}}(2k+1)^{-2s+2} = 1 + \sum_{k\in\mathbb{N}}(2k)^{-2s+2}\left(1 + \frac{1}{2}k^{-1}\right)^{-2s+2} \\
=&1 + \sum_{k\in\mathbb{N}}(2k)^{-2s+2}\Bigg(1 + \frac{-2s+2}{2}k^{-1} + \frac{(-2s+2)(-2s+1)}{8}k^{-2} \\
&+ \frac{(-2s+2)(-2s+1)(-2s)}{48}k^{-3} \\
&+ \frac{(-2s+2)(-2s+1)(-2s)(-2s-1)}{4!2^4}k^{-4} \\
&+ \sum_{v=5}^{\infty}\binom{-2s+2}{v}(2k)^{-v}\Bigg).
\end{aligned}$$

The last equality is due to the generalized binomial theorem. Let $\zeta(s)$ be the Riemann zeta function, then

$$\begin{aligned}\zeta_N^1(s) =& 1 + 2^{-2s+2}\bigg(\zeta(2s-2) - \frac{2s-2}{2}\zeta(2s-1) + \frac{(2s-2)(2s-1)}{8}\zeta(2s) \\ &- \frac{(2s-2)(2s-1)2s}{48}\zeta(2s+1) \\ &+ \frac{(2s-2)(2s-1)2s(2s+1)}{4!2^4}\zeta(2s+2) + \rho(s)\bigg).\end{aligned}$$

Above $\rho(s)$ is analytic for $s > -2$.

We compute the residues

$$\operatorname{Res}_1 \zeta_N^1(1/2) = 2 \operatorname{Res}_1(0 \cdot \zeta(1)) = 0,$$

and

$$\operatorname{Res}_1 \zeta_N^1(3/2) = 2^{-1} \operatorname{Res}_1 \zeta(1) = \frac{1}{2},$$

and

$$\operatorname{Res}_1 \zeta_N^1(-1/2) = 0.$$

Furthermore,

$$\zeta_N^1(-1/2) = 1 + 2^3\left(\zeta(-3) + \frac{3}{2}\zeta(-2) + \frac{6}{8}\zeta(-1) + \frac{6}{48}\zeta(0) - \frac{1}{64}\right) = -\frac{7}{120}.$$

By (3.33), the constant term is

$$b = \frac{1}{32}.$$

In this example we considered a four-dimensional manifold $M$ such that the logarithmic term in the heat trace expansion on $M$ is equal to zero and the constant term $b$ is non-zero.

### 4.2.3 Example $N = T^3$

Consider a four-dimensional manifold $(M, g)$ with one conic singularity and the following metric near the singularity

$$g_{\text{conic}} = dr^2 + r^2 g_N,$$

where $(N, g_N)$ is the three-dimensional flat torus $T^3 = S^1 \times S^1 \times S^1$. We show that the logarithmic term in the heat trace expansion on $M$ is non-zero and try to compute the constant term $b$.

By (4.5),

$$c = -\frac{1}{720}(4\pi)^{-2}\int_{T^3}(5\cdot 36)\,\mathrm{dvol}_{T^3} = -\frac{1}{32\pi^2}\,\mathrm{vol}(T^3)\neq 0.$$

Now we try to compute the constant term

$$\begin{aligned} b = &-\frac{1}{2}\operatorname{Res}_0\zeta_N^1(-1/2) + \frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}}\operatorname{Res}_1\zeta_N^1(-1/2) \\ &-\frac{1}{4}\sum_{j=1}^{m/2} j^{-1}B_{2j}\operatorname{Res}_1\zeta_N^1(j-1/2). \end{aligned}$$

By [BGM, Prop. B.I.2], the eigenvalues of the Laplace operator on the flat torus are

$$\{4\pi^2|n|^2,\ \text{where } n=(n_1,n_2,n_3)\in\mathbb{Z}^3\},$$

therefore

$$\zeta_N^1(s) = \sum_{\lambda_k\in Spec\Delta_N}(\lambda_k+1)^{-s} = \sum_{n_1,n_2,n_3\in\mathbb{Z}}\left(4\pi^2(n_1^2+n_2^2+n_3^2)+1\right)^{-s}. \quad (4.9)$$

On the right hand side we have the Epstein function, which satisfies the following recurrence relation [E, p.78-79]. We use the notation from the book

$$E_k(s;a_1,\dots,a_k,c) := \sum_{n_1,\dots,n_k\in\mathbb{Z}}\left(a_1n_1^2+\cdots+a_kn_k^2+c\right)^{-s},$$

then

$$\begin{aligned} E_k(s;a_1,\dots,a_k,c) =& \sqrt{\frac{\pi}{a_k}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_{k-1}(s;a_1,\dots,a_{k-1},c) \\ &+\frac{4\pi^s}{\Gamma(s)}a_k^{-s/2-1/4}\sum_{n_1,\dots n_{k-1}\in\mathbb{Z}}\left(\sum_{j=1}^{k-1}a_jn_j^2+c\right)^{-s/2+1/4}\times \\ &\times\sum_{n_k=1}^{\infty}n_k^{s-1/2}K_{s-1/2}\left(\frac{2\pi n_k}{\sqrt{a_k}}\sqrt{\sum_{j=1}^{k-1}a_jn_j^2+c}\right). \end{aligned} \quad (4.10)$$

By (4.9) and (4.10), we have

$$
\begin{aligned}
\zeta_N^1(s) =& E_3(s;4\pi^2,4\pi^2,4\pi^2,1) \\
=& \sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_2(s;4\pi^2,4\pi^2,1) \\
&+ \frac{4\pi^s}{\Gamma(s)}(4\pi^2)^{-s/2-1/4}\times \\
&\times \sum_{n_1,n_2\in\mathbb{Z}} \left(4\pi^2 n_1^2 + 4\pi^2 n_2^2 + 1\right)^{-s/2+1/4} \times \\
&\times \sum_{n_3=1}^{\infty} n_3^{s-1/2} K_{s-1/2}\left(n_3\sqrt{4\pi^2 n_1^2 + 4\pi^2 n_2^2 + 1}\right),
\end{aligned}
$$

denote the second summand by $E_3^3(s)$. Then

$$
\zeta_N^1(s) = \sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_2(s;4\pi^2,4\pi^2,1) + E_3^3(s). \tag{4.11}
$$

Use the recurrence formula (4.10) for $E_2(s;4\pi^2,4\pi^2,1)$, to get

$$
\begin{aligned}
E_2(s;4\pi^2,4\pi^2,1) =& \sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_1(s;4\pi^2,1) \\
&+ \frac{4\pi^s}{\Gamma(s)}(4\pi^2)^{-s/2-1/4}\sum_{n_1\in\mathbb{Z}}\left(4\pi^2 n_1^2+1\right)^{-s/2+1/4} \times \\
&\times \sum_{n_2=1}^{\infty} n_2^{s-1/2} K_{s-1/2}\left(n_2\sqrt{4\pi^2 n_1^2+1}\right).
\end{aligned} \tag{4.12}
$$

Put (4.12) into (4.11), to obtain

$$
\begin{aligned}
\zeta_N^1(s) =& \sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_1(s;4\pi^2,1) \\
&+ \sqrt{\frac{1}{4\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{4\pi^s}{\Gamma(s)}(4\pi^2)^{-s/2-1/4}\sum_{n_1\in\mathbb{Z}}\left(4\pi^2 n_1^2+1\right)^{-s/2+1/4} \times \\
&\times \sum_{n_2=1}^{\infty} n_2^{s-1/2} K_{s-1/2}\left(n_2\sqrt{4\pi^2 n_1^2+1}\right) + E_3^3(s),
\end{aligned}
$$

denote the first summand by $E_3^1(s)$ and the second summand by $E_3^2(s)$, to get

$$\zeta_N^1(s) = E_3^1(s) + E_3^2(s) + E_3^3(s).$$

Note that $\Gamma(s-1/2)$ has poles at $s = -j + 1/2$, where $j \in \mathbb{N}$. Now we find poles of $E_3^1(s)$ and $E_3^2(s)$.

We compute using the generalised binomial theorem

$$\begin{aligned}
E_1(s; 4\pi^2, 1) = &\sum_{n_1\in\mathbb{Z}} \left(4\pi^2 n_1^2 + 1\right)^{-s} = \sum_{n_1\in\mathbb{Z}} (4\pi^2 n_1^2)^{-s} \left(1 + (4\pi^2 n_1^2)^{-1}\right)^{-s} \\
=&1 + \sum_{n_1\in\mathbb{N}} 2(2\pi n_1)^{-2s} \Bigg(1 - s(4\pi^2 n_1^2)^{-1} + \frac{-s(-s-1)}{2}(4\pi^2 n_1^2)^{-2} \\
&+ \frac{-s(-s-1)(-s-2)}{6}(4\pi^2 n_1^2)^{-3} + \sum_{v=4}^{\infty} \binom{s}{v} (4\pi^2 n_1^2)^{-v}\Bigg).
\end{aligned}$$

Let $\zeta(s)$ be the Riemann zeta function. Then

$$\begin{aligned}
E_1(s; 4\pi^2, 1) =&1 + \sum_{n_1\in\mathbb{N}} (2\pi)^{-2s} \Bigg(2\zeta(2s) - \frac{s}{2\pi^2}\zeta(2s+2) \\
&+ \frac{s(s+1)}{16\pi^4}\zeta(2s+4) - \frac{s(s+1)(s+2)}{4\cdot 48\pi^6}\zeta(2s+6) \\
&+ \sum_{l=4}^{\infty} (-1)^l C_l \zeta(2s+2l)\Bigg),
\end{aligned}$$

where $C_l > 0$. Since the Riemann zeta function has only one simple pole at 1, $E_1(s; 4\pi^2, 1)$ has simple poles at $s = 1/2 - l$, where $l \in \mathbb{N}_0$.

By the same technique compute

$$\begin{aligned}
\sum_{n_1\in\mathbb{Z}} \left(4\pi^2 n_1^2 + 1\right)^{-s/2+1/4} =& \sum_{n_1\in\mathbb{Z}} (4\pi^2 n_1^2)^{-s/2+1/4} \left(1 + (4\pi^2 n_1^2)^{-1}\right)^{-s/2+1/4} \\
=&1 + 2\sum_{n_1\in\mathbb{N}} (2\pi n_1)^{-s+1/2} \times \\
&\times \Bigg(1 + \left(-\frac{s}{2} + \frac{1}{4}\right)(4\pi^2 n_1^2)^{-1} \\
&+ \frac{(-\frac{s}{2}+\frac{1}{4})(-\frac{s}{2}+\frac{1}{4}-1)}{2}(4\pi^2 n_1^2)^{-2} \\
&+ \sum_{v=3}^{\infty} \binom{-\frac{s}{2}+\frac{1}{4}}{v} (4\pi^2 n_1^2)^{-v}\Bigg),
\end{aligned}$$

hence

$$\begin{aligned}\sum_{n_1\in\mathbb{Z}}\left(4\pi^2 n_1^2+1\right)^{-s/2+1/4} =&1+\sum_{n_1\in\mathbb{N}}(2\pi)^{-s+1/2}\Bigg(2n_1^{-s+1/2}-\frac{2s-1}{8\pi^2}n_1^{-s+1/2-2}\\ &+\frac{(2s-1)(2s+3)}{256\pi^4}n_1^{-s+1/2-4}\\ &+\sum_{l=3}^{\infty}(-1)^l B_l n_1^{-s+1/2-2l}\Bigg)\\ =&1+(2\pi)^{-s+1/2}\Bigg(2\zeta(s-1/2)-\frac{2s-1}{8\pi^2}\zeta(s+3/2)\\ &+\frac{(2s-1)(2s+3)}{256\pi^4}\zeta(s+7/2)\\ &+\sum_{l=3}^{\infty}(-1)^l B_l\zeta(s-1/2+2l)\Bigg),\end{aligned}$$

where $B_l > 0$. Since the Riemann zeta function has only one simple pole, $\sum_{n_1\in\mathbb{Z}}\left(4\pi^2 n_1^2+1\right)^{-s/2+1/4}$ has only simple poles at $s = -2l + 1/2$, where $l \in \mathbb{N}_0$.

To compute the constant term $b$, besides the residues of $\zeta_N^1(s)$, we need to compute the finite part $\operatorname{Res}_0 \zeta_N^1(-1/2)$, i.e. $\operatorname{Res}_0 E_3^1(-1/2)$, $\operatorname{Res}_0 E_3^1(-1/2)$ and $\operatorname{Res}_0 E_3^1(-1/2)$. The above computations show that these are given by the infinite sums involving Riemann zeta function, therefore here we cannot find clear and nice expression for $\operatorname{Res}_0 \zeta_N^1(-1/2)$, consequently we cannot find a nice expression for the constant term in the heat trace expansion in this case.

This example illustrates that if the logarithmic term in the heat trace expansion on $(M, g)$ is non-zero, then it is harder to find a nice expression for the constant term $b$.

## 4.3 Criterion for the logarithmic term to vanish

In this section we develop a geometric condition that is imposed on the cross-section manifold $(N, g_N)$ by the vanishing of the logarithmic term in the heat trace expansion on $(M, g)$.

Let $(N, g_N)$ be a closed manifold of dimension $n$. As before (3.27), we denote the coefficients in the heat trace expansion on $(N, g_N)$ by $a_j^N$, $j \geq 0$.

**Lemma 4.6.** *Let $(M, g)$ be an even-dimensional manifold with a conic singularity. The logarithmic term in the heat trace expansion on $(M, g)$ is equal* *to zero if and only if the following equality holds for the heat trace coefficients of the cross-section manifold* $(N, g_N)$

$$a^N_{\frac{n+1}{2}} = \sum_{k=1}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} a^N_{\frac{n+1}{2}-k}. \tag{4.13}$$

*Proof.* By Theorem 1.1 (c), we have

$$c = \frac{1}{4(4\pi)^{\frac{m}{2}}} \Big( - a^N_{\frac{m}{2}} + \sum_{k=1}^{\frac{m}{2}} (-1)^{k+1} \frac{(m-2)^{2k}}{4^k k!} a^N_{\frac{m}{2}-k} \Big).$$

Use $m = n+1$, to obtain

$$c = \frac{1}{4(4\pi)^{\frac{n+1}{2}}} \Big( - a^N_{\frac{n+1}{2}} + \sum_{k=1}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} a^N_{\frac{n+1}{2}-k} \Big).$$

If $n$ is odd, $c = 0$ is equivalent to

$$a^N_{\frac{n+1}{2}} = \sum_{k=1}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2i}}{4^k k!} a^N_{\frac{n+1}{2}-k}.$$

$\square$

# 5 Explicit expressions of the singular terms

In this section we obtain explicit expressions for the logarithmic term and the constant term $b$ in the heat trace expansion on $(M,g)$ in Theorem 1.1 for some particular $n$-dimensional cross-sections $(N,g_N)$.

## 5.1 Spaces with constant sectional curvature

Let $l\in\mathbb{N}$. Denote by $K_i^l$ the coefficients in the following polynomial

$$\prod_{q=0}^{l-1}\left(v^2-q^2\right)=\sum_{i=1}^{l}K_i^l v^{2i}. \tag{5.1}$$

Let $(N,g_N)$ be a closed manifold and let $a_j^N, j\geq 0$ be the coefficients in the heat trace expansion (3.27) on $(N,g_N)$.

**Theorem 5.1** ( [P, Theorem 1.3.1] )**.** *The coefficients in the heat trace expansion on the odd-dimensional unit sphere $(S^n, g_{round})$ are*

$$a_j^{S^n}=(4\pi)^{\frac{n}{2}}\sum_{l=1}^{\frac{n-1}{2}}\frac{(\frac{n-1}{2})^{2j-n+1+2l}\Gamma(l+\frac{1}{2})K_l^{\frac{n-1}{2}}}{(j+l-\frac{n-1}{2})!(n-1)!},$$

*where numbers $K_l^{\frac{n-1}{2}}$ are given by (5.1).*

*In particular,*

$$a_j^{S^5}=\frac{2^{2j-1}(6-j)\pi^3}{3j!},$$

*and*

$$a_j^{S^7}=\frac{3^{2j-6}(16j^2-286j+1215)\pi^4}{5j!}.$$

Let $(M,g)$ be a manifold with a conic singularity and let $(N,g_N)$ be a cross-section near the singularity.

**Theorem 5.2.** *Let $(N,g_N)$ be an $n$-dimensional, $n$-odd, manifold with constant sectional curvature $\kappa$. Then the logarithmic term in the heat trace expansion on $(M,g)$ can be written as the following polynomial in $\kappa$ of degree $\frac{n+1}{2}$*

$$c=\frac{1}{8\sqrt{\pi}}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)}\sum_{k=0}^{\frac{n+1}{2}}(-1)^{k+1}\frac{(n-1)^{2k}}{4^k k!}\sum_{l=1}^{\frac{n-1}{2}}\frac{(\frac{n-1}{2})^{2l-2k+2}\Gamma(l+\frac{1}{2})K_l^{\frac{n-1}{2}}}{(l-k+1)!(n-1)!}\kappa^{\frac{n+1}{2}-k},$$

*where numbers $K_l^{\frac{n-1}{2}}$ are given by (5.1).*

*Proof.* Since $(N,g)$ has constant sectional curvature $\kappa$, the Riemannian curvature tensor is

$$\mathrm{R}_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk}),$$

the Ricci tensor is

$$\mathrm{Ric}_{ij} = \kappa(n-1)g_{ij},$$

the scalar curvature is

$$\mathrm{Scal} = \kappa n(n-1),$$

where $g_{ij}$ is the round metric on the unit sphere $S^n$.

Consider the heat trace expansion (3.27) on $(N, g_N)$. Since the curvature is constant, every $a_j^N$ in the heat trace expansion is expressed as an integral over $(N, g_N)$ of a constant. By [BGM, Lemma E.IV.5],

$$a_j^N = \kappa^j a_j^{S^n} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)}, \tag{5.2}$$

where $a_j^{S^n}$ is the coefficient in the heat trace expansion on $(S^n, g_{\mathrm{round}})$ and $\mathrm{vol}(N)$ is the volume of $(N, g_N)$.

By Theorem 1.1 (c), the logarithmic term in the heat trace expansion is

$$c = \frac{1}{4(4\pi)^{\frac{n+1}{2}}} \sum_{k=0}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} a^N_{\frac{n+1}{2}-k}.$$

Use (5.2), to obtain

$$c = \frac{1}{4(4\pi)^{\frac{n+1}{2}}} \sum_{k=0}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)} a^{S^n}_{\frac{n+1}{2}-k} \kappa^{\frac{n+1}{2}-k}. \tag{5.3}$$

By Theorem 5.1,

$$a_j^{S^n} = (4\pi)^{\frac{n}{2}} \sum_{l=1}^{\frac{n-1}{2}} \frac{(\frac{n-1}{2})^{2j-n+1+2l}\Gamma(l+\frac{1}{2})K_l^{\frac{n-1}{2}}}{(j+l-\frac{n-1}{2})!(n-1)!},$$

hence

$$a^{S^n}_{\frac{n+1}{2}-k} = (4\pi)^{\frac{n}{2}} \sum_{l=1}^{\frac{n-1}{2}} \frac{(\frac{n-1}{2})^{2l-2k+2}\Gamma(l+\frac{1}{2})K_l^{\frac{n-1}{2}}}{(l-k+1)!(n-1)!}. \tag{5.4}$$

It remains to put (5.4) into (5.3) to obtain the desired expression for the logarithmic term

$$c = \frac{1}{8\sqrt{\pi}} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)} \sum_{k=0}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} \sum_{l=1}^{\frac{n-1}{2}} \frac{(\frac{n-1}{2})^{2l-2k+2}\Gamma(l+\frac{1}{2})K_l^{\frac{n-1}{2}}}{(l-k+1)!(n-1)!} \kappa^{\frac{n+1}{2}-k}.$$

$\square$

The coefficients in the polynomial in Theorem 5.2 might seem complicated. In order to better understand which geometrical information we obtain from Theorem 5.2, we consider some low-dimensional cases.

**Corollary 5.3.** *Let $(N, g_N)$ be a three-dimensional manifold with constant sectional curvature $\kappa$. The logarithmic term in the heat trace expansion on $(M, g)$ is zero if and only if $\kappa = 1$.*

*Proof.* By [CW, Section 9], the coefficients in the heat trace expansion on $S^3$ are

$$a_j^{S^3} = \frac{2\pi^2}{j!}.$$

We compute the first three coefficients

$$a_0^{S^3} = 2\pi^2, \qquad a_1^{S^3} = 2\pi^2, \qquad a_2^{S^3} = \pi^2. \tag{5.5}$$

By (5.3),

$$\begin{aligned} c =& \frac{1}{4(4\pi)^2} \frac{\operatorname{vol}(N)}{\operatorname{vol}(S^3)} \sum_{k=0}^{2} (-1)^{k+1} \frac{2^{2k}}{4^k k!} a_{2-k}^{S^3} \kappa^{2-k} \\ =& - \frac{1}{4(4\pi)^2} \frac{\operatorname{vol}(N)}{\operatorname{vol}(S^3)} \left( a_2^{S^3} \kappa^2 - a_1^{S^3} \kappa + \frac{1}{2} a_0^{S^3} \right). \end{aligned}$$

Using (5.5), we obtain

$$c = - \frac{1}{4(4\pi)^2} \frac{\operatorname{vol}(N)}{\operatorname{vol}(S^3)} \pi^2 (\kappa^2 - 2\kappa + 1).$$

Finally, the logarithmic term $c = 0$ if and only if $\kappa = 1$. □

**Corollary 5.4.** *Let $(N, g_N)$ be a five-dimensional manifold with constant sectional curvature $\kappa$. The logarithmic term in the heat trace expansion on $(M, g)$ is zero if and only if one of the following is true $\kappa = 1$ or $\kappa = 2$.*

*Proof.* We compute the logarithmic term in the heat trace expansion. By Theorem 5.1, we compute

$$a_0^{S^5} = \pi^3, \qquad a_1^{S^5} = \frac{10\pi^3}{3}, \qquad a_2^{S^5} = \frac{16\pi^3}{3}, \qquad a_3^{S^5} = \frac{16\pi^3}{3}. \tag{5.6}$$

By (5.3),

$$\begin{aligned} c =& \frac{1}{4(4\pi)^3} \frac{\operatorname{vol}(N)}{\operatorname{vol}(S^5)} \sum_{k=0}^{3} (-1)^{k+1} \frac{4^{2k}}{4^k k!} a_{3-k}^{S^3} \kappa^{3-k} \\ =& - \frac{1}{4(4\pi)^3} \frac{\operatorname{vol}(N)}{\operatorname{vol}(S^5)} \left( a_3^{S^3} \kappa^3 - 4 a_2^{S^3} \kappa^2 + 8 a_1^{S^3} \kappa - \frac{32}{3} a_0^{S^3} \right). \end{aligned}$$

Using (5.6), we obtain

$$c = -\frac{1}{4(4\pi)^3}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^5)}\frac{16\pi^3}{3}\left(\kappa^3 - 4\kappa^2 + 5\kappa - 2\right).$$

Since $\kappa^3 - 4\kappa^2 + 5\kappa - 2 = (\kappa-1)^2(\kappa-2)$, the logarithmic term $c = 0$ if and only if one of the following is true $\kappa = 1$ or $\kappa = 2$. $\square$

Finally, we apply Theorem 5.2 in the seven-dimensional case.

**Corollary 5.5.** *Let $(N, g_N)$ be a seven-dimensional manifold with constant sectional curvature $\kappa$. The logarithmic term in the heat trace expansion on $(M, g)$ is zero if and only if one of the following is true $\kappa = 1$ or $\kappa = \frac{225}{109} \pm \frac{36\sqrt{5}}{109}$.*

*Proof.* We compute the logarithmic term in the heat trace expansion. By Theorem 5.1, we compute

$$a_0^{S^7} = \frac{5}{3}\frac{\pi^4}{5}, \qquad a_1^{S^7} = \frac{35}{3}\frac{\pi^4}{5}, \qquad a_2^{S^7} = \frac{707}{18}\frac{\pi^4}{5},$$
$$a_3^{S^7} = \frac{167}{2}\frac{\pi^4}{5}, \qquad a_4^{S^7} = \frac{3\cdot 327}{8}\frac{\pi^4}{5}.$$

By (5.3),

$$\begin{aligned} c =& \frac{1}{4(4\pi)^4}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^7)}\sum_{k=0}^{4}(-1)^{k+1}\frac{3^{2k}}{k!}a_{4-k}^{S^7}\kappa^{4-k} \\ =& -\frac{1}{4(4\pi)^4}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^7)}\left(a_4^{S^7}\kappa^4 - 9A_3^{S^7}\kappa^3 + \frac{3^4}{2}a_2^{S^7}\kappa^2 - \frac{3^5}{2}a_1^{S^7}\kappa + \frac{3^7}{8}a_0^{S^7}\right). \end{aligned}$$

We use the precise expressions for the heat trace expansion coefficients on $S^7$, to obtain

$$\begin{aligned} c =& -\frac{1}{4(4\pi)^4}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^7)}\frac{\pi^4}{5}\left(\frac{3\cdot 327}{8}\kappa^4 - \frac{9\cdot 167}{2}\kappa^3 + \frac{3^2\cdot 707}{4}\kappa^2\right. \\ & \left. - \frac{3^4\cdot 35}{2}\kappa + \frac{3^6\cdot 5}{8}\right) \\ =& -\frac{1}{4(4\pi)^4}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^7)}\frac{\pi^4}{5}\frac{9}{8}\left(109\kappa^4 - 668\kappa^3 + 1414\kappa^2 - 1260\kappa + 405\right). \end{aligned}$$

Since $109\kappa^4 - 668\kappa^3 + 1414\kappa^2 - 1260\kappa + 405 = (\kappa-1)^2(109\kappa^2 - 450\kappa + 405)$, the logarithmic term $c$ is zero if and only if one of the following is true $\kappa = 1$ or $\kappa = \frac{225}{109} \pm \frac{36\sqrt{5}}{109}$. $\square$

Summing up, in this section we consider an $n+1$-dimensional manifold $(M, g)$ with a conic singularity such that the cross-section $(N, g_N)$ near the singularity has constant sectional curvature, and assume that the logarithmic term in the heat trace expansion on $(M, g)$ vanishes. While in the four-dimensional case the only possible cross-section $(N, g_N)$ is a spherical space form, in the six-dimensional case besides a spherical space form we may have a cross-section with sectional curvature $\kappa = 2$. Furthermore, in the eight-dimensional case besides spherical space forms we may have cross-sections with even more peculiar sectional curvatures $\kappa = \frac{225}{109} \pm \frac{36\sqrt{5}}{109}$.

## 5.2 The logarithmic and the constant term for $N = S^n$

In this section we show that the logarithmic term $c$ and the constant term $b$ in the heat trace expansion on $(M, g)$ for $N = S^n$ with the round metric, both are equal to zero.

First, we prove a more general result. We consider spherical space forms.

**Lemma 5.6.** *Let $(M, g)$ be a manifold with conic singularities and let the cross-section manifold near every singularity be isometric to some spherical space form. Then the logarithmic term in the heat trace expansion on $(M, g)$ is equal to zero.*

*Proof.* We consider one cross-section $(N, g_N)$, which is isometric to a spherical space form, and prove that its contribution to the logarithmic term in the heat trace expansion on $(M, g)$ is zero. If every cross-section is isometric to a spherical space form, we conclude that the sum of all contributions is zero as desired.

Assume that $(M, g)$ is odd-dimensional, by Theorem 1.1 (c) we obtain $c = 0$.

Assume now that $(M, g)$ is even-dimensional. By (5.2), the coefficients $a_j^N$ in the heat trace expansion on $(N, g_N)$ are proportional to the coefficients in the heat trace expansion on $S^n$

$$a_j^N = a_j^{S^n} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)}. \tag{5.7}$$

We use Theorem 1.1 (c) and (5.7), to obtain

$$c = \frac{1}{4(4\pi)^{\frac{n+1}{2}}} \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)} \sum_{k=0}^{\frac{n+1}{2}} (-1)^{k+1} \frac{(n-1)^{2k}}{4^k k!} a^{S^n}_{\frac{n+1}{2}-k}. \tag{5.8}$$

Therefore the logarithmic term $c$ for $(N, g_{\mathrm{round}})$ differs from the logarithmic term $c_{S^n}$ for $(S^n, g_{\mathrm{round}})$ by the factor $\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)}$. Assume that we prove that

$c_{S^n} = 0$, then it follows that the logarithmic term $c$ is zero for any spherical space form.

Now we prove that the logarithmic term $c_{S^n}$ in the heat trace expansion on $(M, g)$ for $N = S^n$ is zero.

Observe,

$$\frac{(n+k-2)(n+k-3)\dots(n+1)n}{k!} = \frac{(n+k-2)(n+k-3)\dots(k+1)}{(n-1)!}. \tag{5.9}$$

By Proposition 4.4 and (5.9), we obtain

$$\begin{aligned}
&\zeta_{S^n}^{\frac{n-1}{2}}(s) \\
&\quad = \sum_{k\geq 0} \frac{(n+k-2)(n+k-3)\dots(k+1)}{(n-1)!}(n+2k-1)\left(k+\frac{n-1}{2}\right)^{-2s} \\
&\quad = \sum_{k\geq 0} \frac{2(n+k-2)(n+k-3)\dots(k+1)}{(n-1)!}\left(k+\frac{n-1}{2}\right)^{-2s+1}.
\end{aligned}$$

Let $n = 2u+1$ and $k+u = l$, then

$$\begin{aligned}
\zeta_{S^n}^{\frac{n-1}{2}}(s) &= \sum_{k\geq 0} \frac{2(2u+k-1)(2u+k-2)\dots(k+1)}{(2u)!}(k+u)^{-2s+1} \\
&= \sum_{l\geq u} \frac{2(l+u-1)(l+u-2)\dots(l-u+1)}{(2u)!} l^{-2s+1} \\
&= \sum_{l\geq u} \frac{2\prod_{q=1}^{u-1}(l^2-q^2)}{(2u)!} l^{-2s+2} = \frac{2}{(2u)!}\sum_{l\geq 1} l^{-2s+2}\prod_{q=1}^{u-1}(l^2-q^2).
\end{aligned}$$

The last equality is true, because $\sum_{l=1}^{u-1}\prod_{q=1}^{u-1}(l^2-q^2) = 0$.

Let $K_i^u$ be defined by (5.1). Then

$$\begin{aligned}
\zeta_{S^n}^{\frac{n-1}{2}}(s) &= \frac{2}{(2u)!}\sum_{l\geq 1} l^{-2s}\sum_{i=1}^{u} K_i^u l^{2i} \\
&= \frac{2}{(2u)!}\sum_{l\geq 1}\sum_{i=1}^{u} K_i^u l^{-2s+2i}.
\end{aligned}$$

Finally

$$\zeta_{S^n}^{\frac{n-1}{2}}(s) = \frac{2}{(2u)!}\sum_{i=1}^{u} K_i^u \zeta(2s-2i), \tag{5.10}$$

where $\zeta(s)$ is the Riemann zeta function. We compute

$$\mathrm{Res}_1\, \zeta_{S^n}^{\frac{n-1}{2}}(-1/2) = \frac{2}{(2u)!}\sum_{i=1}^{u} K_i^u\, \mathrm{Res}_1\, \zeta(-1-2i) = 0,$$

because the Riemann zeta function has no poles at points $-1-2i$ for $i = 1, \dots, u$. We conclude that in the case of spherical space form $(N, g_{\text{round}})$, the logarithmic term in the heat trace expansion on $(M, g)$ is zero

$$c = \frac{1}{4}\frac{\mathrm{vol}(N)}{\mathrm{vol}(S^n)} Res_1 \zeta_{S^n}^{\frac{n-1}{2}}(-1/2) = 0.$$

$\square$

**Lemma 5.7.** *Let $(M, g)$ be a manifold with conic singularities and let the cross-section manifold near every singularity be isometric to the odd-dimensional unit sphere $(S^n, g_{round})$. Then the constant term $b$ in the heat trace expansion on $(M, g)$ is equal to zero.*

*Proof.* We consider one cross-section $(N, g_{\text{round}})$ and prove that its contribution to the constant term $b$ in the heat trace expansion on $(M, g)$ is zero. If every cross-section is isometric to $(S^n, g_{\text{round}})$, we conclude that the sum of all contributions is zero as desired.

By (5.10),

$$\zeta_{S^n}^{\frac{n-1}{2}}(s) = \frac{2}{(n-1)!}\sum_{i=1}^{\frac{n-1}{2}} K_i^{\frac{n-1}{2}} \zeta(2s-2i),$$

using Theorem 1.1 (b), we obtain

$$\begin{aligned} b = &-\mathrm{Res}_0\, \frac{1}{(n-1)!}\sum_{i=1}^{\frac{n-1}{2}} K_i^{\frac{n-1}{2}} \zeta(-1-2i) \\ &- \frac{1}{2}\sum_{j=1}^{\frac{n+1}{2}} j^{-1} B_{2j} \frac{1}{(n-1)!}\sum_{i=1}^{\frac{n-1}{2}} K_i^{\frac{n-1}{2}}\, \mathrm{Res}_1\, \zeta(2j-1-2i). \end{aligned}$$

Since $\operatorname{Res}_1 \zeta(2j-1-2i)$ is non-zero only for $2j-1-2i=1$, we get the relation $j=i+1$. Using

$$\zeta(-d) = (-1)^d \frac{B_{d+1}}{d+1}, \qquad d \in \mathbb{N},$$

we get

$$b = \frac{1}{(n-1)!} \sum_{i=1}^{\frac{n-1}{2}} K_i^{\frac{n-1}{2}} \frac{B_{2i+2}}{2i+2} - \frac{1}{2} \sum_{i=1}^{\frac{n-1}{2}} (i+1)^{-1} B_{2i+2} \frac{1}{(n-1)!} K_i^{\frac{n-1}{2}} = 0,$$

as desired. □

## 5.3 The logarithmic and the constant terms for $N = \mathbb{R}P^n$

In this section we consider a manifold $(M,g)$ with a conic singularity with the cross-section $N = \mathbb{R}P^n$ with the round metric. We show that in this case the logarithmic term in the heat trace expansion on $(M,g)$ is equal to zero. We compute the constant term $b$ and show that it is not equal to zero.

**Theorem 5.8.** *Let $(M,g)$ be a manifold with a conic singularity with the cross-section $N = \mathbb{R}P^n$, where $n = 4v+1$, $v \in \mathbb{N}$. Then the logarithmic term in the heat trace expansion is zero and the constant term $b$ is not equal to zero*

$$b = \sum_{i=1}^{\frac{n-1}{2}} \left(2^{2i+2} - 1\right) \frac{1}{4(i+1)(n-1)!} B_{2i+2} K_i^{\frac{n-1}{2}} \neq 0.$$

*Above $B_j$ are Bernoulli numbers and $K_j^n \in \mathbb{R}$ are given by (5.1).*

*Proof.* Since the projective space is a spherical space form, the first claim of the theorem follows from Lemma 5.6.

By Proposition 4.5, the multiplicity of the $k$-th eigenvalue of $\mathbb{R}P^n$ is

$$\begin{aligned}
\mu_k =& \frac{(2k+n-2)!}{(2k)!(n-1)!}(n+4k-1) \\
=& \frac{(2k+n-2)(2k+n-3)\dots(2k+1)}{(n-1)!}(4k+n-1).
\end{aligned}$$

Therefore

$$\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(s) = \sum_{k\geq 0} \frac{(2k+n-2)(2k+n-3)\dots(2k+1)}{(n-1)!}(4k+n-1)\times$$

$$\times\left(2k+\frac{n-1}{2}\right)^{-2s}.$$

Let $n = 4v+1$ and $k+v = l$. Then

$$\begin{aligned}
\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(s) &= \sum_{k\geq 0} \frac{(2k+4v-1)(2k+4v-2)\dots(2k+1)}{(4v)!} 2\,(2k+2v)^{-2s+1} \\
&= \sum_{l\geq v} \frac{(2l+2v-1)(2l+2v-2)\dots(2l-2v+1)}{(4v)!} 2\,(2l)^{-2s+1} \\
&= \sum_{l\geq v} \frac{\left((2l)^2-(2v-1)^2\right)\dots\left((2l)^2-1\right)\cdot 2l}{(4v)!} 2^{-2s+2} l^{-2s+1} \\
&= \sum_{l\geq v} \frac{\prod_{q=1}^{2v-1}\left((2l)^2-q^2\right)}{(4v)!} 2^{-2s+3} l^{-2s+2}.
\end{aligned}$$

The expression $\sum_{l=1}^{v-1}\prod_{q=1}^{2v-1}\left((2l)^2-q^2\right) = 0$, because in every summand one of the terms $(2l)^2 - q^2$ in the product is equal to zero. Therefore, we can add this to $\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(s)$ and it remains unchanged. We obtain

$$\begin{aligned}
\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(s) &= \sum_{l\geq 1} \frac{\prod_{q=0}^{2v-1}\left((2l)^2-q^2\right)}{(4v)!} 2^{-2s+1} l^{-2s} \\
&= \sum_{l\geq 1}\sum_{i=1}^{2v} K_i^{2v}(2l)^{2i}\frac{2^{-2s+1}}{(4v)!} l^{-2s} \\
&= \sum_{l\geq 1}\sum_{i=1}^{2v} \frac{2^{-2s+2i+1}}{(4v)!} K_i^{2v} l^{-2s+2i} \\
&= \sum_{i=1}^{2v} \frac{2^{-2s+2i+1}}{(4v)!} K_i^{2v}\zeta(2s-2i).
\end{aligned}$$

Above $\zeta(s)$ is the Riemann zeta function, which has only one simple pole at $s = 1$. In particular, $\zeta(s)$ has no pole at $-2i-1$ for $i = 1,\dots,2v$.

Consequently, $\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(s)$ is analytic at $s = -1/2$. By

$$\zeta(-d) = (-1)^d \frac{B_{d+1}}{d+1}, \qquad d \in \mathbb{N},$$

we obtain

$$\begin{aligned}
\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(-1/2) &= \sum_{i=1}^{2v} \frac{2^{2i+2}}{(4v)!} K_i^{2v} \zeta(-2i-1) \\
&= \sum_{i=1}^{2v} \frac{2^{2i+2}}{(4v)!} K_i^{2v} (-1)^{2i+1} \frac{B_{2i+2}}{2i+2},
\end{aligned}$$

and simplify

$$\zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(-1/2) = -\sum_{i=1}^{2v} \frac{2^{2i+1}}{(i+1)(4v)!} B_{2i+2} K_i^{2v}. \tag{5.11}$$

To find the constant term $b$, it remains to compute the following sum of the residues

$$\begin{aligned}
&\sum_{j=1}^{2v+1} j^{-1} B_{2j} \operatorname{Res}_1 \zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(j-1/2) \\
=& \sum_{j=1}^{2v+1} j^{-1} B_{2j} \sum_{i=1}^{2v} \frac{2^{-2j+2i+2}}{(4v)!} K_i^{2v} \operatorname{Res}_1 \zeta(2j-2i-1).
\end{aligned}$$

The summand in the above expression is not equal to zero only for $2j-2i-1 = 1$, equivalently $j = i+1$. We obtain

$$\sum_{j=2}^{2v+1} j^{-1} B_{2j} \operatorname{Res}_1 \zeta_{\mathbb{R}P^n}^{\frac{n-1}{2}}(j-1/2) = \sum_{i=1}^{2v} \frac{1}{(i+1)(4v)!} B_{2i+2} K_i^{2v}. \tag{5.12}$$

Using (5.11) and (5.12), we compute

$$
\begin{aligned}
b =& -\frac{1}{2}\zeta^{\frac{n-1}{2}}_{\mathbb{R}P^n}(-1/2) - \frac{1}{4}\sum_{j=1}^{2v+1} j^{-1} B_{2j} \operatorname{Res}_1 \zeta^{\frac{n-1}{2}}_{\mathbb{R}P^n}(j-1/2) \\
=& \frac{1}{2}\sum_{i=1}^{2v} \frac{2^{2i+1}}{(i+1)(4v)!} B_{2i+2} K_i^{2v} - \frac{1}{4}\sum_{i=1}^{2v} \frac{1}{(i+1)(4v)!} B_{2i+2} K_i^{2v} \\
=& \sum_{i=1}^{2v} \left(2^{2i} - \frac{1}{4}\right) \frac{1}{(i+1)(4v)!} B_{2i+2} K_i^{2v} \\
=& \sum_{i=1}^{\frac{n-1}{2}} \left(2^{2i+2} - 1\right) \frac{1}{4(i+1)(n-1)!} B_{2i+2} K_i^{\frac{n-1}{2}}.
\end{aligned}
$$

Note that by construction the numbers $K_i^{\frac{n-1}{2}}$ are alternating as $i$ grows, the Bernoulli numbers $B_{2i+2}$ are also alternating. We conclude that all products $B_{2i+2}E_i^{\frac{n-1}{2}}$ have the same sign and $b \neq 0$. $\square$

## 5.4 The logarithmic term for $N = T^n$

In this section we consider an even-dimensional manifold $(M, g)$ with a conic singularity with the crosssection $N = T^n = S^1 \times \cdots \times S^1$ with the flat metric. We compute the logarithmic term in the heat trace expansion on $M$ and show that it is not equal to zero.

**Proposition 5.9** ( [BGM, Prop.B.I.2, p.148]). *Let $(T^n, g_{flat})$ be an $n$-dimensional torus with the flat metric. The Laplace-Beltrami operator has the following eigenvalues on $T^n$*

$$
\lambda = \sum_{i=1}^{n} 4\pi^2 k_i^2, \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n,
$$

*the multiplicity of each eigenvalue is equal to one.*

**Theorem 5.10.** *Let $(M, g)$ be a manifold with a conic singularity with the cross-section $N = T^n$, where $n \geq 3$ is odd. Then the logarithmic term in the heat trace expansion on $(M, g)$ is*

$$
c = \frac{(-1)^{\frac{n+1}{2}} (n-1)^{n+1}}{2^{2n+4}\pi^{n+1/2}(\frac{n+1}{2})!} \neq 0.
$$

*Proof.* The shifted zeta function for the flat torus is

$$\zeta_{T^n}^{\frac{n-1}{2}}(s) = \sum_{k\in\mathbb{Z}^n} \left(4\pi^2 k_1^2 + \ldots 4\pi^2 k_n^2 + \left(\frac{n-1}{2}\right)^2\right)^{-s},$$

which is the Epstein zeta function. Since $n$ is odd, by [K, pp. 319–320], $\zeta_{T^n}^{\frac{n-1}{2}}(s)$ is a meromorphic function with the simple poles at the points $s = \frac{n}{2}, \ldots, 1, -\frac{1}{2}, -\frac{2l+1}{2}, l \in \mathbb{N}$ with the residues

$$Res_1 \zeta_{T^n}^{\frac{n-1}{2}}(s) = \frac{(-1)^{n/2+s}\pi^{s/2}\left(\frac{n-1}{2}\right)^{n-2s}}{\sqrt{(4\pi^2)^n}\Gamma(s)\Gamma(\frac{n}{2}-s+1)}.$$

Compute the residue at $s = -1/2$

$$\begin{aligned} Res_1 \zeta_{T^n}^{\frac{n-1}{2}}(-1/2) =& \frac{(-1)^{n/2-1/2}\pi^{-1/4}\left(\frac{n-1}{2}\right)^{n+1}}{\sqrt{(4\pi^2)^n}\Gamma(-\frac{1}{2})\Gamma(\frac{n}{2}+\frac{1}{2}+1)} = \frac{(-1)^{\frac{n-1}{2}}\pi^{-1/4}\left(\frac{n-1}{2}\right)^{n+1}}{(2\pi)^n(-2\sqrt{\pi})\Gamma(\frac{n+3}{2})} \\ =& \frac{(-1)^{\frac{n+1}{2}}(n-1)^{n+1}}{2^{n+2}(2\pi)^n\pi^{1/2}(\frac{n+1}{2})!} = \frac{(-1)^{\frac{n+1}{2}}(n-1)^{n+1}}{2^{2n+2}\pi^{n+1/2}(\frac{n+1}{2})!}. \end{aligned}$$

The expression above is non-zero for $n \geq 3$. By (3.31), the logarithmic term in the heat trace expansion is

$$c = \frac{(-1)^{\frac{n+1}{2}}(n-1)^{n+1}}{2^{2n+4}\pi^{n+1/2}(\frac{n+1}{2})!}.$$

$\square$

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